This is an excerpt from Lial/Miller's "Intermediate Algebra", 5th edition, Scott, Foresman and Company, Glenview, Illinois, 1988 pages 36 – 42

Chapter 2

Linear Equations and Inequalities

2.1 Linear Equations in One Variable

Objectives
1. Define linear equations.
2. Solve linear equations by:
   a. Simplifying;
   b. Using the addition and multiplication properties;
   c. Using the distributive property.

An algebraic expression is the result of performing the basic operations of addition, subtraction, multiplication, and division (except by 0), or extraction of roots on any collection of variables and numbers. Some examples of algebraic expressions include

$8x + 9$, $\sqrt{y} + 4$, $\frac{x^3 y^8}{z}$.

1. Applications of mathematics often lead to equations, statements that two algebraic expressions are equal. A linear equation in one variable involves only real numbers and one variable. Examples include

   $x + 1 = -2$, $y - 3 = 5$, and $2k + 5 = 10$.

   Linear Equation: An equation is linear if it can be written in the form

   $ax + b = c$,

   where a, b, and c are real numbers, with $a \neq 0$.

   A linear equation is also called a first-degree equation, since the highest power on the variable is one.

   If the variable in an equation is replaced by a real number that makes the statement true, then that number is a solution of the equation. For example, 8 is a solution of the equation $y - 3 = 5$, since replacing y with 8 gives a true statement. An equation is solved by finding its solution set, the set of all solutions. The solution set of the equation $y - 3 = 5$ is {8}.

2. The first step in solving a linear equation is to use the properties of Chapter 1 to simplify each side of the equation as much as possible.

[S.K. note: The properties being referred to here are the
Properties of equality
Properties of inequality
Properties of the Real Numbers; Closure, Commutative, Associative, Distributive, Identity, Inverse properties of addition and multiplication.
Multiplication by zero
Double negative property
Order of Operations]

The next two examples show how expressions are simplified by using these properties.

Example 1. Simplify each expression.

a. $4x + 8x = (4 + 8)x = 12x$
b. $3r – 7r = (3 – 7)r = -4r$
c. $12m + m = 12m + 1m = 13m$
d. $y + y = 1y + 1y = 2y$
e. $5p + 7q + 4$ cannot be simplified further.

The expressions 5p, 7q, and 4 of Example 1e are examples of terms. A term is a number or the product of a number and one or more variables. Terms with exactly the same variables raised to exactly the same powers are called like terms.

The expression $-5a + 3 – 7a + 15a – 8$ can be simplified by using the commutative and associative properties to rearrange the terms so that like terms are together. Then like terms can be combined by using the distributive property.

$$-5a + 3 – 7a + 15a – 8 = -5a – 7a + 15a + 3 – 8$$
$$= (-5 – 7 + 15)a + 3 – 8$$
$$= 3a – 5$$

The distributive property also can be used to simplify $3(4k + 1) = 3(4k) + 3(1) = 12k + 3$.

Example 2. Simplify the expression.

$5y – 8y – 6y + 11y$

$= (5 – 8 – 6 + 11)y$
$= 2y$

Simplifying expressions as in Example 2 is called combining like terms. Only like terms may be combined.

Math Wiz companion text to this excerpt:

It's difficult to overemphasize the importance of linear equations. Why? Because straight lines are always easier to deal with than other shapes, so we prefer to use them. A plumber has a much simpler task if he runs a straight pipe from a hot water heater to the hot water inlet of your tub than he does if he has to route that pipe around ducts or wall supports.

But there are applications for lines that you might not expect. Here's one example. Electrical Engineers have found that it's easiest to design electronic equipment using components that behave linearly. A component is anything that produces an expected change to a signal. That signal might be something like voltage or current or light intensity. If the component behaves linearly, that means the signal coming out is some set multiple of the signal going in. (By multiple, don't forget that that includes multiplying by fractions, too, so that the output could be smaller than the input. It could also be negative).

An example of this would be a radio signal amplifier. In a radio broadcast station, the sound from the announcer's voice, the music, the ads are all converted to low-frequency radio waves, then overlaid on what's called the carrier signal – the central frequency assigned to that station for broadcasting. That combined signal has to be amplified in power so that it can be detected by the distant radios in your house or car.

The amplifier is usually a linear device. If the signal going into it is 1 watt, the signal coming out is going to be some known multiple, perhaps 100 times. So the output signal will be 100 watts for an input signal of 1 watt. What makes the amplifier linear is that some other input power level, say 10 watts, will also be multiplied by 100 times. The new output power will be 1000 watts.

A linear equation that describes this amplifier is actually predicting what the output power will be for any given input. In an algebraic equation, we'd call the input and the output the variables of the equation. Notice that there's one variable you can change – the input – and a second variable that you can't – the output. The output depends entirely on the component's multiplication factor. The variable you can tweak (the input) is called the independent variable in algebra. The variable that's predetermined by the component (the output) is called the dependent variable.

By tradition, we usually label the independent variable "x" and the dependent variable "y". So a linear equation would be written in the form

$y = Ax$ (y, the output = some multiple of x, the input.)

This form of equation would describe almost any line you can imagine, because the constant multiple represented by A could be any real number, ½, 3, -7, even zero.

Let's verify that this form that we're calling a linear equation actually does represent a line. We'll use the examples of y = ½ x, and y = 3x. We'll calculate y for a few selected values of x.

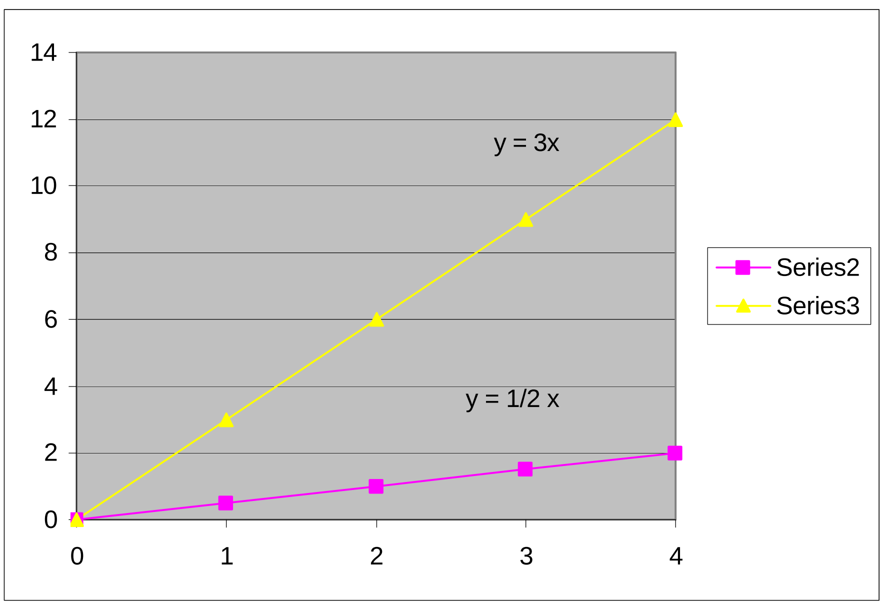
y = ½ x

x	y = ½ x
0	0
1	1/2
2	1
4	2

y = 3x

x	y = 3x
0	0
1	3
2	6
3	9

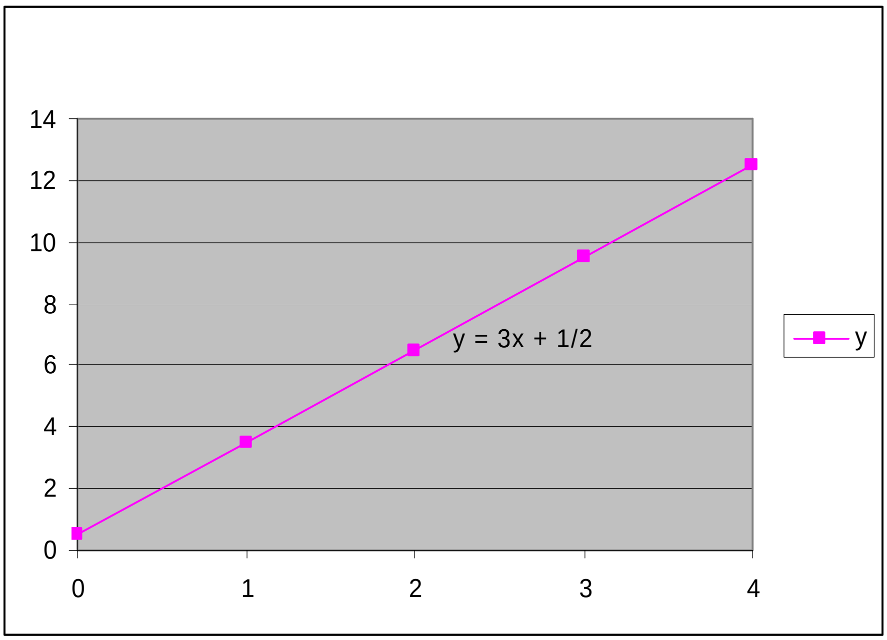
Let's graph these:



Sure enough. The two equations both describe lines. They also describe the behavior of the linear amplifier; they predict the amplifier's output (the y value) for any given input (the x value).

To expand our definition of a linear equation to include all possible lines, we have to take one more step. Notice that both of our example lines start at x = 0, y = 0. But what if our amplifier does something unexpected at zero? Maybe we'd like to be able to offset our line to represent that. Let's use y = 3x and pretend that when x = 0 (there's no input power), we see that our amplifier "leaks" a little current and outputs what appears to be a tiny signal. And, that little extra signal is always being leaked. Then, for each value of x, y would equal 3x + that leaked signal. We'll say it's 1/2 watt. How can our line represent this?

Since, for every value of x, y = 3x + the leaked signal and the leaked signal = 1/2 watt, the equation is just what you might expect; y = 3x + 1/2. We've added a constant offset to the line. Look how this shifts the whole line up by ½ watt:



It's a bit subtle, but if you look at the corner of the graph at the x = 0,y = 0 point, you can see the line doesn't cross right though (0,0). It crosses through (0, ½).

So now we're able to use equations to describe any linear behavior.

If we want to know what a particular output value is for some input, or what the input value must be for a given output, we say we are seeking a solution to that linear equation. For instance, in our example of y = 3x + ½, if we wonder what the output value would be for an input value of x = 1, we're also asking what the solution of the equation is for x = 1.

Let's use this background information to set down some formal definitions.

Term: a number or a product of a number and one or more variables. Examples: 3, 2x, -4xy. The terms of our example linear equation are y, 3x, and ½.

Algebraic expression: Performing mathematical operations on terms. These operations include addition, subtraction, multiplication, division (except by 0), or extraction of roots. (Division by 0 is excluded because division by zero is undefined according to traditional mathematics.) The algebraic expressions of our example linear equation are y and 3x + ½.

Equation: A statement that two algebraic expressions are equal. Thus, y = 3x + ½ is an equation. The fact that it describes a line makes it a linear equation.

All lines are generally described by the form y = Ax + b.
x represents the independent variable,
y represents the dependent variable,
A represents the multiplicative constant,
b represents the offset constant and may equal zero.

Let's say you have a component and you want to find out if its behavior is linear. That would mean collecting many observations, then extracting an equation that fits those observations. In general, you'd apply many different inputs and measure each corresponding output, then use trial and error to figure out the equation (or use software or advanced methods).

In any case, you could find yourself with a complicated expression that you'd like to simplify. A few examples:

a. 4x + 8x = (4 + 8)x = 12x
b. 3r – 7r = (3 – 7)r = -4r
c. 12m + m = 12m + 1m = 13m
d. y + y = 1y + 1y = 2y
e. 5p + 7q + 4 cannot be simplified further.

What does this imply? That you can only combine (add or subtract) terms that represent the same thing. If you have 3 measurements of the independent variable x, you may combine them to simplify the expression. But combining, say, x terms with constant terms is not logical. An input signal is not the same as a component's current leak. Formally, we say that we may only combine "like terms".

Let's go back to our favorite example:

y = 3x + ½
If you had an original equation of:

5y – 4y = x + 2x + 1 – ½
It would simplify to:
(5 – 4)y = (1 + 2)x + (1 – ½)
1y = 3x + ½

$y = 3x + ½$

In a linear equation, both variables are of the first power (no exponent). Later, when dealing with more complex equations, some variables will be raised to powers. In those cases, like terms also require that the variable's powers match. So you can combine:

$x^2 + 3x^2$

but you can't combine

$x^3 + 3x^2$.

As long as you combine only like terms, you can use the associative, distributive, and commutative laws to simplify expressions and equations. Associative and commutative laws allow you to group like terms together. The distributive law allows like terms to be combined. Some examples:

Example 1.
$-5a + 3 – 7a + 15a – 8$
$= -5a – 7a + 15a + 3 – 8$
$= (-5 – 7 + 15)a + 3 – 8$
$= 3a – 5$

Example 2.
$3(4k + 1)$
$= 3(4k) + 3(1)$
$= 12k + 3$.

Example 3.
$5y – 8y – 6y + 11y$
$= (5 – 8 – 6 + 11)y$
$= 2y$