

# MATHCOUNTS

2002

■ National Competition ■  
Sprint Round  
Problems 1-30

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1. Since the set includes negative numbers, asking for the smallest product is the same as asking for the negative product with the largest magnitude. The only way to get a negative product from the provided set of numbers is to choose 1 negative number and 2 positive numbers. To get the product with the largest magnitude, choose the negative number with the largest magnitude ( $-1$ )

and choose the two positive numbers with the largest magnitudes  $\left(\frac{5}{4} \text{ and } 1\right)$ . Compute

$$-1 \times \frac{5}{4} \times 1 = -\frac{5}{4}. \text{ The answer is } -\frac{5}{4}.$$

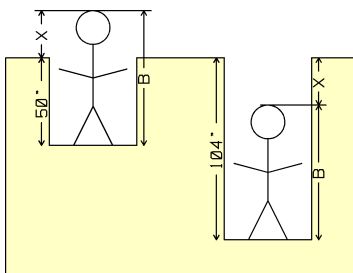
2. One way to approach this problem is to factor the number 17160. Then we'll see if we can spot 4 consecutive integers, from the factors.  $17160 = 2 \times 2 \times 2 \times 3 \times 5 \times 11 \times 13$ . We will guess that 11 and 13 are two of the four consecutive numbers. Therefore, 12 must also be one of the numbers. Since  $12 = 2 \times 2 \times 3$ , the only factors remaining are 2 and 5. We now know that  $10 \times 11 \times 12 \times 13 = 17160$ .  $10 + 11 + 12 + 13 = 46$ , which is the answer.

Another approach is to find a single number, which raised to the fourth power would be approximately 17160. Then we can use guess-and-check on nearby consecutive integers to find the correct answer. To estimate the fourth root, I'll take the square root of the square root of 17160. Since calculators are not permitted, I estimate the square root of 17160 is a little larger than 130. ( $130 \times 130 = 16900$ ) Next I note that the square root of 130 is larger than 11 and less than 12. ( $11 \times 11 = 121$ .  $12 \times 12 = 144$ .) For my first guess, I will assume that 11 and 12 are the middle two integers. Since  $10 \times 11 \times 12 \times 13 = 17160$ , our first guess is proven correct. The sum of these integers is 46.

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3. Draw a picture!



My sketch contains two parts. The first part shows Becca (whose height is “B”) standing in a 50” deep hole. Her head is “X” inches above the ground. The second part shows Becca standing in a 104” deep hole. The top of her head is “X” inches below the ground. Once the illustration is done, you can use algebra, “guess and check”, or logical reasoning to solve the problem.

Using logical reasoning, Becca’s feet have descended 54 inches, so the top of her head has also descended 54 inches. But the illustration also shows her head has descended “X” inches to ground level and another “X” inches below ground level. So “X” must be half of 54 inches. From the first part of the illustration, Becca’s height is  $50+X = 50+27 = 77$  inches! Becca is unusually tall, so we should check our work using the second part of the illustration.  $104-X = 104-27 = 77$  inches.

4. There are 12 perfect squares from 1 to 150: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, and 144. There are 5 perfect cubes from 1 to 150: 1, 8, 27, 64, and 125. The numbers 1 and 64 are common to both sets. (These are the only two perfect sixth powers from 1 to 150.) Together, there are  $12+5-2=15$  perfect squares/cubes in the bag. This is  $1/10$  of the total. The probability that the number drawn from the bag is neither a perfect square nor a perfect cube is  $9/10$ . This is the answer.

Note that if you knew the numbers common to both sets, it wouldn’t really be necessary to list all of the perfect squares or cubes. It would only be necessary to **count** all of the perfect squares and cubes, and deduct the number of common numbers.

5. If we cross-multiply the given equation, the result is:

$$5(10x - 3y) = 3(13x - 2y)$$

$$50x - 15y = 39x - 6y$$

If we gather the  $x$  terms on one side of the equation and the  $y$  terms on the other side, we’ll be able to solve for  $x/y$ :

$$50x - 15y = 39x - 6y$$

$$50x - 39x = 15y - 6y$$

$$11x = 9y$$

$$\frac{x}{y} = \frac{9}{11}$$

This is the ratio of  $x$  to  $y$ .

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6. My approach on this problem is to obtain the prime factorization of 1560.  $1560=2 \times 2 \times 2 \times 3 \times 5 \times 13$ . No value of  $n!$  less than  $13!$  can contain a factor of 13. Also note that  $2 \times 2 \times 2 \times 3 \times 5 = 120 = 5!$ . Therefore, any value of  $n!$  greater than or equal to 5 will contain all of the other factors of 1560. The answer is 13.
7. At every node, except the start and finish, the pencil must draw an arc (line segment) entering and an arc leaving the node. Therefore, every node, except the start and finish must have an even number of arcs touching the node. Nodes C, D, E, and F are touched by 4 arcs. Node G is touched by 2 arcs. Node A is touched by 3 arcs. And node B is touched by 5 arcs. If the tracing begins at A, the only possible place it could end is at B.
8. "Guess and Check" is a good strategy for this problem. Note the divisibility rule that any number that is divisible by 9 has a sum-of-digits that are a multiple of 9. Also note the divisibility rule that any number that is divisible by 5 has a units digit of 0 or 5. The largest 3-digit numbers that are one greater than a multiple of 9 are 991, 982, 973, 964, 955, 946, 937, 928, 919, and 910. The largest number from this set that is 3 greater than a multiple of 5 is 973. The answer is 973.
9. If the radio station randomly selects any date in December, the contestant will win. There are 31 days in December, so there are 31 chances of a win. The contestant will also win if the radio station selects the 31st of any other month. Seven months have 31 days: January, March, May, July, August, October, and December. Since December 31 has already been counted, there are 6 additional days that will win a prize for the contestant. There are  $31+6=37$  ways for the contestant to win out of 366 possible choices. The answer is  $37/366$ .
10. Multiply the inequality by  $\pi$ . Then substitute the estimate,  $\pi=3.14$  for  $\pi$ :

$$\begin{aligned} -5\pi &\leq x \leq 10\pi \\ -15.7 &\leq x \leq 31.4 \end{aligned}$$

The smallest integer value of  $x$  that satisfies this inequality is  $-15$ . The largest integer value is 31. The problem asks for the sum of all integer solutions. The sum of values from  $-15$  to  $+15$  will be zero. We're left with finding the sum of the integers from 16 to 31. There are many methods to calculate this sum. One method is to multiply the number of elements in the sequence (16) by the average value in the sequence (23.5). The sum of a set of 16 numbers, whose average value is 23.5 is  $16 \times 23.5 = 376$ . The answer is 376.

11. A cube has 6 faces, 8 corners, and 12 edges. If we start with 12 edges, and slice off a little bit of a corner, we will add 3 additional edges. Repeat for each corner, and we will end with  $12+8 \times 3 = 36$  edges on the final polyhedron.
- Alternative solution: The problem states that each face of the original cube will have an octagonal shape after slicing is complete. The problem also states that each corner will expose a new face, with a triangular shape. The total number of sides visible on these 2-dimensional faces are: 6 original faces  $\times$  8 sides per face = 48 sides; plus 8 new faces  $\times$  3 sides per face = 24 sides. Altogether, there are  $48+24=72$  sides. An edge of a 3-dimensional polyhedron occurs where two faces meet each other. So there will be  $72 \div 2=36$  edges in the final polyhedron.
12. Over the course of 3 hours, the hour hand will travel one quarter of a circle. The minute hand will make three full circles. If the hands were the same length, the ratio of the distances traveled would be 1:12. However, the hour hand is 6 units long and the minute hand is 8 units long, so the ratio of the distances traveled by the tips of the hands will be 6:96, which reduces to 1:16. The answer, expressed as a common fraction, is  $1/16$ .
13. How many 1-digit perfect squares are there? There are 3 of them (1, 4, and 9). How many 2-digit perfect squares are there? There are 9 perfect squares less than 100, but we've already listed 3 of them. There are 6 2-digit-perfect squares (16, 25, 36, 49, 64, and 81). If we follow the procedure in the question, we will have written 15 digits. The 16th through 18th digits are the 10th perfect square (100). The 19th through 21st digits are the 11th perfect square (121). The 40th through 42nd digits are the 18th perfect square (324). The 40th digit is 3.

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14. The equation can be solved at  $x=0, y=24$  and also at  $y=0, x=36$ . If graph paper is available, you could draw a line segment from the point  $(0, 24)$  to the point  $(36, 0)$ . Including the endpoints, you could count the number of lattice points (points where both  $x$  and  $y$  are integers) that fall along the line. Without graph paper, I would consider making a table:

x	y	$2x+3y$	Comments
0	24	72	Solution
1	23.33	72	Not a solution. $y$ is not a whole number.
1.5	23	72	Not a solution. $x$ is not a whole number.
2	22.67	72	Not a solution. $y$ is not a whole number
3	22	72	Solution
4	21.33	72	Not a solution. $y$ is not a whole number
5	20.67	72	Not a solution. $y$ is not a whole number
6	20	72	Solution
9	18	72	Solution
12	16	72	Solution
15	14	72	Solution
18	12	72	Solution
21	10	72	Solution
24	8	72	Solution
27	6	72	Solution
30	4	72	Solution
33	2	72	Solution
36	0	72	Solution

The first few rows of the table show that  $y$  is an integer only where  $x$  is a multiple of 3, and  $x$  is an integer only where  $y$  is a multiple of 2. There are 13 rows of the table where  $x$  and  $y$  are non-negative integers (whole numbers). The first value that works is the ordered pair  $(0, 24)$ . The last value that works is the ordered pair  $(36, 0)$ . The answer is 13.

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15. We are given the lengths of the 3 sides, in terms of a parameter,  $a$ . We can set up a quadratic equation:

$$(3a-1)+(a^2+1)+(a^2+2)=16$$

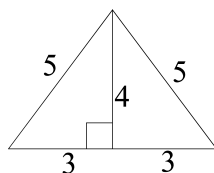
$$2a^2+3a+2=16$$

$$2a^2+3a-14=0$$

Application of the quadratic formula will yield one positive integer solution. Alternatively, you might attempt “guess and check” to see if there is an easy solution:

$a$	$3a-1$	$a^2+1$	$a^2+2$	$(3a-1)+(a^2+1)+(a^2+2)$
0	-1	1	2	2
1	2	2	3	7
2	5	5	6	16

The value  $a=2$  provides a solution – an isosceles triangle, with side lengths of 5, 5, and 6. The isosceles triangle is shown:



The isosceles triangle can be divided into two right triangles. Each of these is a 3–4–5 right triangle. Now that we know the height of the isosceles triangle, the area of the 5–5–6 triangle is

$$A = \frac{1}{2}bh = \frac{1}{2} \times 6 \times 4 = 12.$$

You can also use Heron’s formula for the area of an arbitrary triangle. Given the lengths of the sides,  $s_1$ ,  $s_2$ , and  $s_3$  and the semiperimeter,  $s$ :

$$A = \sqrt{s \times (s - s_1) \times (s - s_2) \times (s - s_3)}$$

$$A = \sqrt{8 \times (8 - 5) \times (8 - 5) \times (8 - 6)}$$

$$A = \sqrt{8 \times 3 \times 3 \times 2}$$

$$A = \sqrt{144}$$

$$A = 12$$

16. Every factor of a number has a corresponding cofactor. The product of the factor and the cofactor is equal to the number itself. The problem asks how many factors are larger than 7!. Since  $8! = (7!) \times 8$ , this problem is equivalent to asking how many cofactors are smaller than 8. Every integer from 1 to 8, inclusive is a factor of 8!. Therefore, every integer smaller than 8 is also a factor of 8!. The answer is 7.

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17. I apply the “difference of squares” identity. The statement of the problem tells me the difference of the squares of the smallest two numbers. The statement of the problem also tells me the numbers are consecutive odd integers, so I know the difference between the first two numbers is 2:

$$b^2 - a^2 = (b - a) \times (b + a)$$

$$344 = 2 \times (b + a)$$

$$172 = b + a$$

The average of  $a$  and  $b$  are  $172 \div 2 = 86$ . Since  $a$ ,  $b$ , and  $c$  are consecutive odd numbers,  $a=85$ ,  $b=87$ , and  $c=89$ . Once again, I apply the difference-of-squares identity:

$$c^2 - b^2 = (c - b) \times (c + b)$$

$$c^2 - b^2 = 2 \times 176$$

$$c^2 - b^2 = 352$$

The answer is 352.

18. I will rewrite the problem to solve for  $n!$ . Then cancel matching terms. Then rewrite the remaining product in factorial order:

$$n! = \frac{10!}{3 \times 5!}$$

$$n! = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)}$$

$$n! = 10 \times 9 \times 8 \times 7$$

$$n! = (2 \times 5) \times (3 \times 3) \times (2 \times 2 \times 2) \times 7$$

$$n! = 2 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 7$$

$$n! = 7 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 2$$

$$n! = 7 \times (2 \times 3) \times 5 \times (2 \times 2) \times 3 \times 2$$

$$n! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$n! = 7!$$

The answer is 7.

19. The problem is equivalent to asking how many numbers, less than 100, are multiples of 6, and have no prime factors, except 2 and 3. The largest integer less than  $100 \div 6 = 16$ . Which natural numbers less than or equal to 16 have no prime factors, except 2 and/or 3? These are 1, 2, 3, 4, 6, 8, 9, 12, and 16. There are 9 natural numbers that meet the conditions of the problem.

Alternatively, we can make a 2-dimensional “multiplication” table, showing possible factors of 2 in one dimension and possible factors of 3 in the other dimension:

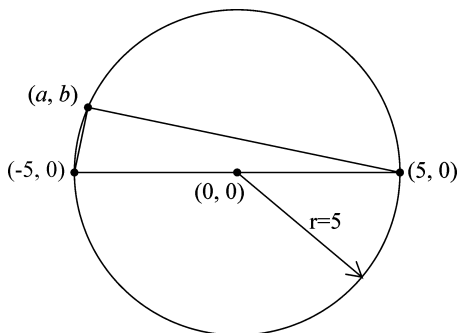
$\times$	$3^1=3$	$3^2=9$	$3^3=27$	$3^4=81$
$2^1=2$	6	18	54	$\geq 100$
$2^2=4$	12	36	$\geq 100$	$\geq 100$
$2^3=8$	24	72	$\geq 100$	$\geq 100$
$2^4=16$	48	$\geq 100$	$\geq 100$	$\geq 100$
$2^5=32$	96	$\geq 100$	$\geq 100$	$\geq 100$
$2^6=64$	$\geq 100$	$\geq 100$	$\geq 100$	$\geq 100$

The table shows there are 9 values, less than 100, for which the product of the prime factors is 6. This is the answer.

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20. The first clause of the problem requires that point  $(a, b)$  lie a distance 5 units from the origin. The second clause of the problem requires the area of the triangle must be 10 square units. I draw a sketch:



The locus of points 5 units from the origin is a circle of radius 5, centered at the origin. The two fixed points of the given triangle give it a base of 10 units. From the formula for the area of a triangle,  $A = \frac{1}{2}bh$ , the area of the triangle will be 10 square units if the height of the triangle is 2 units. This occurs only when  $b = \pm 2$ . Both conditions of the problem are met where the horizontal line  $y = +2$  intersects the circle or where the horizontal line  $y = -2$  intersects the circle. There are 4 such solutions, so the answer is 4.

21. The statement is a logic statement of the form “if P then Q”. This means that if condition P is true, then Q must also be true. If condition P is false, there is no requirement on the truth of Q.

This given logic statement is equivalent to the contrapositive logic statement, “if not Q then not P”. This means that if condition Q is false, then P must also be false. If condition Q is true, there is no requirement on the truth of P.

In the case of the cards shown, every card containing a vowel must be turned over to prove the statement of the problem. (The card containing the letter A must be turned over, but the cards containing the letters B and C do not have to be turned over.)

Considering the contrapositive, every card **not** containing a prime number must also be turned over to complete the proof of the problem statement. (The card containing the number 4 must be turned over, but the cards containing the numbers 3 and 5 do not have to be turned over.)

The answer is 2.

22. There is only one way in which Dora can walk clockwise around the shaded region in 4 moves. At each intersection, the probability that Dora walks in the correct direction is  $\frac{1}{4}$ . So the

probability that Dora will make the correct choice at each intersection is:  $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256}$

Similarly, the probability that Dora will walk counterclockwise around the shaded region in 4 moves is  $\frac{1}{256}$ . The problem permits Dora to walk in either direction around the shaded region, so

the answer is  $\frac{1}{256} + \frac{1}{256} = \frac{1}{128}$ .

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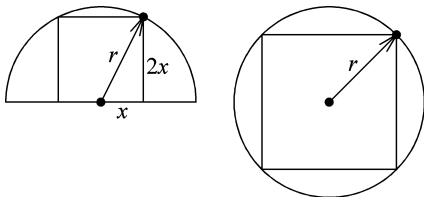
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23. Rewrite and simplify the expression:

$$\begin{aligned} 12!+14! &= 12!(1+13\times 14) \\ &= 12\times(1+13\times 14) \\ &= 12\times(1+182) \\ &= 12\times 183 \\ &= 12\times 3\times 61 \end{aligned}$$

The largest prime factor of  $12!$  is 11. The largest prime factor of the overall expression is 61.

24. The inscribed squares are shown in the following illustration:

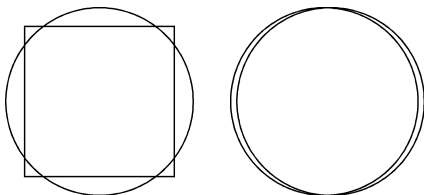


For the semicircle, we begin with the Pythagorean Theorem, then solve for the area, in terms of the radius,  $r$ . For the circle, we use the formula for the area of a square, given the length of the diagonal.

$$\begin{aligned} r^2 &= x^2 + (2x)^2 \\ r^2 &= 5x^2 \\ r &= \sqrt{5}x \\ x &= \frac{1}{\sqrt{5}}r \\ 2x &= \frac{2}{\sqrt{5}}r \\ (2x)^2 &= \frac{4}{5}r^2 = A \end{aligned} \qquad \begin{aligned} A &= \frac{d^2}{2} \\ A &= \frac{(2r)^2}{2} \\ A &= \frac{4r^2}{2} \\ A &= 2r^2 \end{aligned}$$

Now, the ratio of the area of the square inscribed in the semicircle to the area of the area of the square inscribed in the circle is  $\frac{4}{5}r^2 : 2r^2$ . This reduces to  $\frac{4}{5} : 2 = 4 : 10 = 2 : 5 = \frac{2}{5}$ .

25. The maximum number of points of intersection of a circle and a square is 8. The maximum number of points of intersection of two distinct circles is 2. Examples of these simple two-figure intersections are shown:



Since we are asked about two circles and a square, we might have as many as  $8+8+2$  points of intersection. However, we must convince ourselves that we can really get all of these intersections. Begin with the circle and square shown on the left, above. Imagine a second circle, of the same size drawn a very small distance to the right of the first circle. Clearly, we will have 8 distinct intersections with the square. And clearly, we will have 2 intersections between the circles. The maximum number of intersections is 18.

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26. By setting pairs of set elements equal to each other, and solving for  $n$ , we can find plausible values of  $n$  for which there are only 3 distinct elements of the set. Since the number of combinations of 4 things taken 2 at a time is 6, there are 6 equations to be solved:

$n - 2 = n + 2$		$n - 2 = \frac{n}{2}$
$0 = 4$	$n - 2 = 2n$	$2n - 4 = n$
No solution	$-2 = n$	$n = 4$
	$n + 2 = \frac{n}{2}$	$2n = \frac{n}{2}$
$n + 2 = 2n$	$2n + 4 = n$	$4n = n$
$2 = n$	$n = -4$	$3n = 0$
		$n = 0$

Since each of the 5 possible values of  $n$  are integers, and since each is unique, we know the solution to the problem is 5. We could check our work by computing each of the five sets  $*(n)$

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27. Performing a long-division of the numerator polynomial by the denominator polynomial yields a quotient of  $x+5$ , with a remainder of 20. In other words, we can rewrite the numerator polynomial as follows:

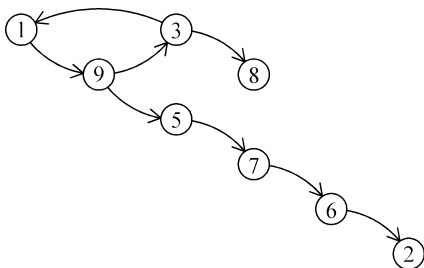
$$\begin{aligned} x^2 + 2x + 5 &= (x-3)(x+5) + 20 \\ &= (x^2 + 2x - 15) + 20 \\ &= x^2 + 2x + 5 \end{aligned}$$

Now, we substitute the rewritten numerator polynomial:

$$\begin{aligned} \frac{x^2 + 2x + 5}{x-3} &= \frac{(x-3)(x+5) + 20}{x-3} \\ &= \frac{(x-3)(x+5)}{x-3} + \frac{20}{x-3} \\ &= (x+5) + \frac{20}{x-3} \end{aligned}$$

For integer values of  $x$ , the first term is always an integer. The complete quotient will be an integer where the second term is an integer. This occurs where  $x-3 \in \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20\}$ . The largest value of  $x$  occurs where  $x-3$  is largest. When  $x-3=+20$ ,  $x=23$ . The answer is 23.

28. Let's orient the cube, so that blue face is towards the top. Since there are two red faces, we can also orient the cube so that a red face is towards the front. A second red face towards the bottom is one uniquely painted cube. A second red face towards the back is the second uniquely painted cube. A second red face towards the right side is the third uniquely painted cube. A second red face towards the left side is painted the same as the previous case. We have enumerated all possibilities, so the answer is 3.
29. I begin by listing all two-digit multiples of 19 or 31. These are 19, 31, 38, 57, 62, 76, 93, and 95. Then I construct a state diagram. The state diagram begins with the first digit of the 2002-digit number (1). Arcs connect to each of the next possible digits. For example, after the digit 1, the only possible choice is the digit 9. But after the digit 9, we can have either of the digits 3 or 5.

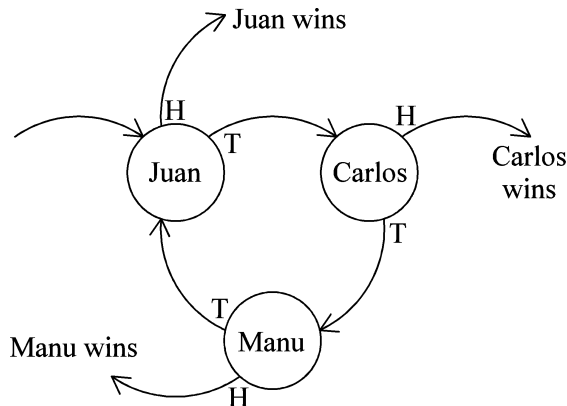


The only loop in the state diagram is between the digits 1, 9, and 3. Until we approach the end of the 2002-digit number, these digits will repeat 1-9-3-1-9-3-1-9-3-.... Note the digit 3 will occur in the 3rd, 6th, 9th, and every other digit position, which is a multiple of 3. If the last 3 occurs in the 2001st position, the final digit of the 2002-digit number can be 1 or 8. If the last 3 occurs in the 1998th position, the final digit of the 2002-digit number must be 7. The answer is 8.

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30. To help think about the problem, sketch a diagram of the game. The diagram below shows how the state of the game progresses from one player to the next, and how wins can be achieved. The circles represent “states”. Juan’s circle, for example, represents the state of the game when it is his turn to flip. The lines and arrows show how the game can move from one state to another. As soon as Juan flips, he may win immediately (“Juan wins”) or play may pass to Carlos. Carlos only gets to flip after Juan and only if Juan doesn’t win:



The diagram above shows Juan starting the game with one flip. Juan always gets to flip for a win once, plus after half of Manu’s flips, on average. Carlos gets to flip for a win after half of Juan’s flips, on average. Manu gets to flip for a win after half of Carlos’ flips, on average. Writing equations for the number of flips, each player gets:

$$\begin{aligned} \text{Juan} &= J = 1 + M/2 \\ \text{Carlos} &= C = J/2 \\ \text{Manu} &= M = C/2 \end{aligned}$$

Substituting these equations into each other, we can find the number of flips each player will receive in an average game:

$$\begin{aligned} M &= J/4 \\ M &= (1 + M/2) / 4 \\ M &= 1/4 + M/8 \\ \frac{7}{8} M &= 1/4 \\ M &= 1/4 \times \frac{8}{7} = \frac{2}{7} \\ J &= 1 + M/2 = 1 + \frac{1}{7} = \frac{8}{7} \\ C &= J/2 = \frac{4}{7} \end{aligned}$$

On average, Juan will get  $1\frac{1}{7}$  flips per game. Carlos will get  $\frac{4}{7}$  flips per game, and Manu will get  $\frac{2}{7}$  flips per game.

The number of flips the players, Juan, Carlos, and Manu receive is in the proportion 8:4:2. With a fair coin, the probability that a player will win is proportional to the number of flips the player gets. Therefore, the probability that Manu will win is 2 out of 14, or  $\frac{1}{7}$ .

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1. This problem has no element of profit or loss. Thus there is no need to compute or to keep track of how much was paid when buying stock or how much was made when selling stock. We are given the final number of shares held, so we don't even need to keep track of how many shares are held. We just need to track the price-per-share of the stock to the end of the problem. The price started as \$20 per share. The price increases by 20% (\$4.00 per share) to \$24.00 per share. The price decreases by 10% (\$2.40 per share) to \$21.60 per share. The price increases by 50% (\$10.80) to \$32.40 per share. What is the value of 30 shares of stock? Multiply the price per share by the number of shares to yield the total value of \$972.00.

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2. In the decimal number system, zeros appears at the end of a positive integer when the integer is divisible by a power of 10. The number of zeros is determined by the largest power of 10 that divides the integer. In a factorial, there are always at least as many factors of 2 as factors of 5. Therefore, if we can determine the number of factors of 5 (the power of 5) that divides the factorial, we know the number of zeros that end the factorial.

Remember that  $2002! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \dots \times 2000 \times 2001 \times 2002$ . We can count the number of factors that contain a factor of 5.

Let's list the numbers which have at least one factor of 5: 5, 10, 15, 20, 25, ..., 2000. Here are 400 factors of 5.

List the numbers which have at least two factors of 5: 25, 50, 75, 100, 125, ..., 2000. Here are 80 more factors of 5.

List the numbers which have three factors of 5: 125, 250, 375, 500, 625, ..., 2000. Here are 16 more factors of 5.

List the numbers which have four factors of 5: 625, 1250, 1875. Here are 3 more factors of 5.

We have found that  $2002!$  contains  $400+80+16+3=499$  factors of 5. So  $2002!$  must end with 499 zeros.

In general, the power of a prime,  $p$ , that divides  $n!$  can be computed by the following formula:

$$\begin{aligned} f(n, p) &= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \\ f(2002, 5) &= \left\lfloor \frac{2002}{5^1} \right\rfloor + \left\lfloor \frac{2002}{5^2} \right\rfloor + \left\lfloor \frac{2002}{5^3} \right\rfloor + \left\lfloor \frac{2002}{5^4} \right\rfloor + \left\lfloor \frac{2002}{5^5} \right\rfloor + \dots \\ &= \left\lfloor \frac{2002}{5} \right\rfloor + \left\lfloor \frac{2002}{25} \right\rfloor + \left\lfloor \frac{2002}{125} \right\rfloor + \left\lfloor \frac{2002}{625} \right\rfloor + \left\lfloor \frac{2002}{3125} \right\rfloor + \dots \\ &= \lfloor 400.4 \rfloor + \lfloor 80.08 \rfloor + \lfloor 16.016 \rfloor + \lfloor 3.2032 \rfloor + \lfloor 0.64064 \rfloor + \dots \\ &= 400 + 80 + 16 + 3 + 0 + \dots \\ &= 499 \end{aligned}$$

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3. Let's consider the 3 horizontal rows, marked with the horizontal arrows. According to the statement of the problem each of these rows must have the same sum, which I will call "s". If we add all three rows together, the sum is 3s. But the problem also states that each of the digits 1 through 9 is used once. We don't know where the digits appear, but we do know that each of these digits is used in one of the three rows. So  $3s=1+2+3+\dots+9$ .  $3s=45$ .  $s=15$ . Now, we know that every marked line must total 15. Any line that contains two numbers leads to a unique number in the remaining position.

$$\begin{array}{ccc} 1 & ? & 8 \\ & ? & ? \\ & 5 & ? \end{array}$$

$1+8=9$ , so the empty cell on the top row must contain 6.  $1+5=6$ , so the first empty cell in the second row must be 9.  $8+5=13$ , so the center cell must contain 2.

$$\begin{array}{ccc} 1 & 6 & 8 \\ & 9 & 2 \\ & 5 & ? \end{array}$$

$9+2=11$ , so the third cell in the second row must contain 4.  $6+2=8$ , so the second cell in the bottom row must contain 7.

$$\begin{array}{ccc} 1 & 6 & 8 \\ & 9 & 2 \\ & 5 & 7 \end{array}$$

$5+7=12$ , so the shaded cell must contain 3. Also  $8+4=12$ , so the shaded cell must contain 3.

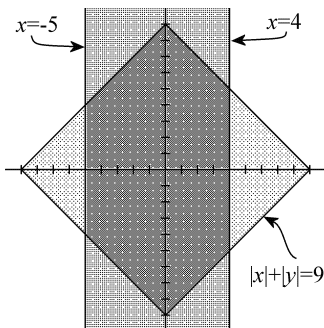
$$\begin{array}{ccc} 1 & 6 & 8 \\ & 9 & 2 \\ & 5 & 7 \end{array}$$

Check that each digit is used exactly once. The answer is 3.

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4. The diamond in the graph below is a graph of the equation,  $|x| + |y| = 9$ . The shaded interior of the diamond represents the inequality,  $|x| + |y| \leq 9$ . The vertical lines,  $x = -5$  and  $x = 4$  are also graphed in the equation. The shaded area between the vertical lines represents the inequality,  $-5 \leq x \leq 4$ . The darkly shaded area of the graph shows the area where both inequalities are true.



The darkly shaded region consists of two trapezoids. The trapezoid on the left of the  $y$ -axis has base lengths of 18 and 8 and a height of 5. The trapezoid on the right of the  $y$ -axis has base lengths of 18 and 10 and a height of 4. The areas of these trapezoids are given by:

$$A = \frac{1}{2} \times (b_1 + b_2) \times h$$

$$A_1 = \frac{1}{2} \times (18 + 8) \times 5$$

$$A_1 = 13 \times 5$$

$$A_1 = 65$$

$$A_2 = \frac{1}{2} \times (18 + 10) \times 4$$

$$A_2 = 14 \times 4$$

$$A_2 = 56$$

$$A = A_1 + A_2$$

$$A = 65 + 56$$

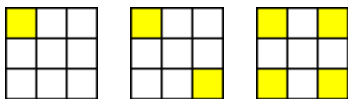
$$A = 121$$

Alternatively, the area of the darkly shaded region can be computed as the area of the diamond (162) less the area of the two lightly shaded isosceles right triangles (16 and 25).  
 $162 - 16 - 25 = 121$ .

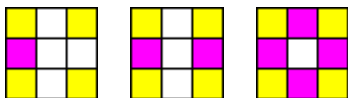
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5. I begin by coloring the upper-left square of the figure. Then I reflect along the diagonal line of symmetry. Then I reflect along the vertical line of symmetry.



Next I color a vacant square along the edge of the figure. Then I reflect along the vertical line of symmetry. Then I reflect along the diagonal line of symmetry.



Finally, I color the vacant center square with a third color.



The maximum possible number of colors I can use is three.

6. Note that the units digits of the pattern of numbers repeats: 1, 3, 5, 7, 9. Also note the sum of these units digits is 25. It happens that 25 evenly divides 50. The row that contains two full repetitions of this pattern is the row we are seeking. This is the row that contains 10 elements, and this is row number 10.
7. Let  $B$  be the number of blue marbles we remove from the first jar. Then  $34-B$  is the number of yellow marbles we remove from the first jar. The number of blue marbles we place in the second jar is  $B$ . The number of yellow marbles remaining in the first jar is  $25-(34-B) = B-9$ . The difference between the number of blue marbles in the second jar and the number of yellow marbles in the first jar is  $(B)-(B-9) = 9$ .

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8. Let's consider the minimum and maximum integer values we might have in the list. Let's postulate that a "5" appears in the list. We can always replace the "5" with "2" and "3". The sum will remain the same, but the product will be "6", which is larger. Therefore, no "5" can appear in the solution list. Similarly, no integer value greater than or equal to 5 can appear in the solution list.

Let's postulate that a "1" appears in the list. We can always combine the "1" with any other number in the list, and replace the two elements by their sum. The sum will remain the same, but the product will be larger. Therefore, no "1" can appear in the solution list.

The problem constraint calls for positive integer values in the list. We have proven this list can only contain the numbers "2", "3", or "4". Now let's consider how multiple instances of these list elements can be replaced with others.

Let's postulate that a pair of 4's appear in the list. The sum is 8 and the product is 16. We can always replace the 4's with 2, 3, and 3. The sum will remain the same, but the product will be "18", which is larger. Therefore, the solution list cannot contain two or more 4's.

Let's postulate that three 2's appear in the list. The sum is 6 and the product is 8. We can always replace three 2's with a pair of 3's. The sum will remain the same, but the product will be "9", which is larger. Therefore, the solution list cannot contain three or more 2's.

Let's postulate that a "2" and a "4" appear in the list. The sum is 6 and the product is 8. We can always replace these two elements with a pair of 3's. The sum will remain the same, but the product will be "9", which is larger. Therefore, the solution list cannot contain both "2" and "4".

At this point, the only solution lists we have not disproven are:

- All 3's. [However, this cannot sum to 20.]
- One 4. All of the rest 3's. [This cannot sum to 20.]
- One 2. All of the rest 3's.
- Two 2's. All of the rest 3's. [This cannot sum to 20.]

There is only one solution that has not been disproven!  $2+3+3+3+3+3+3=20$ .  
 $2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 1458$ . The answer is 1458.

Alternatively, since time is a significant factor, you might consider a solution set comprised of a single value. With twenty 1's, the product is 1. With ten 2's, the product is 1024. With five 4's, the product is 1024. With four 5's, the product is 625. With two 10's, the product is 100. With one 20, the product is 20. My intuition suggests that the maximum will occur between 2 and 4, so my goal is to maximize the number of 3's in the solution.

Consider a similar problem, where the sum of the elements must be 12. (The LCM of 2, 3, and 4.)  $2^6=64$ ,  $3^4=81$ ,  $4^3=64$ . This tends to confirm our intuitive feeling that we should maximize the number of 3's in the solution.

Finally, note the form of the single-valued solutions are always  $a^{20/a}$ . A graphing calculator can find the maximum value of this expression, which occurs at  $a=e=2.71828\dots$   $e^{20/e} \cong 1568.05$  This suggests the best integer values to use will be near 2.718.

# MATHCOUNTS

2002

■ National Competition ■  
Team Round  
Problems 1-10

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1. This map can be colored, region-by-region, without any backtracking. Begin by finding an unlabeled region that is adjacent to 3 different-colored regions. The C-shaped region surrounding the central “2” must be labeled “4”. Similarly, the  $\cap$ -shaped region which touches the “4”, “1”, and “3” must be labeled “2”. Continuing in this matter, the C-shaped region surrounding our first C-shaped region must be labeled “1”. Then the roughly triangular-shaped region to the right of the “3” must be labeled “4”. The curved region, along the top edge, must be labeled “3”. The curved region along the bottom edge, must be labeled “2”. Now we find that region “R” must be labeled “3”. Note that there are still 2 regions left unlabeled, but they are not necessary to solving this problem. The answer is “3”.
2. Let’s assume the small box of popcorn contains 1 unit of volume. The medium box, in which two dimensions have been doubled will contain 4 units of volume. The large box, in which three dimensions have been doubled will contain 8 units of volume. If we add the volume of one small box, one medium box, and one large box of popcorn, we have  $1+4+8=13$  units of volume. At \$0.75 per unit volume, we have  $13 \times \$0.75 = \$9.75$ .

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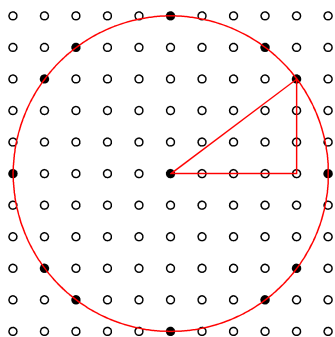
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3. The only trick to this problem is to devise a systematic method of counting the rectangles. By slightly stretching the given figure, it can be redrawn on a lattice grid. Now there are only nine distinct sizes of rectangles, which we must count. I'll list these in a table. As a sanity check, I also list the maximum possible number of rectangles that could be drawn.

size of rectangle (h×w)	maximum possible number of these rectangles drawn on a 3×3 grid	number of rectangles in given figure
1×1	9	7
1×2	6	2
1×3	3	1
2×1	6	5
2×2	4	2
2×3	2	1
3×1	3	3
3×2	2	2
3×3	1	1

There are a total of 24 rectangles.

4. Note the radius of the circle is an integer. Also note that the radius, 5, is the hypotenuse of a 3-4-5 triangle. This assures there will be several lattice points on the circle. The circle is shown below on a lattice grid. The lattice points that lie on the circle are highlighted. The lattice point at the origin is highlighted. A radius (the hypotenuse of a 3-4-5 triangle) is shown.



Most of the lattice points are contained on or inside the circle. In each corner of the lattice grid, there are 10 lattice points that lie outside the circle. Of the 121 lattice points shown, 40 are outside the circle. Therefore, there are 81 lattice points on or inside the circle. The answer is 81.

Note that it isn't necessary to draw an accurate circle. A set of 3 straight line segments connecting the points (5, 0) to (4, 3) to (3, 4) to (0, 5) are sufficient to prove which points lie inside or outside the quarter circle.

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5. The perpendicular bisector of any chord of a circle will pass through the center of the circle. The points  $(-2, 0)$  and  $(8, 0)$  form a chord of the circle. The midpoint of this chord is at  $(3, 0)$ . Since the chord is a horizontal line, the perpendicular bisector is the vertical line,  $x=3$ .

The points  $(-2, 0)$  and  $(0, 2)$  form a chord with a slope of 1. The midpoint of this chord is at the point  $(-1, 1)$ . The perpendicular bisector has a slope of  $-1$ . A few of the lattice points which lie on this perpendicular bisector are  $(-1, 1)$ ,  $(0, 0)$ ,  $(1, -1)$ ,  $(2, -2)$ , and  $(3, -3)$ .

This lattice point lies on the other chord's perpendicular bisector, so the center of the circle is at the point  $(3, -3)$ .

6. The 10-cent and 15-cent denomination stamps enable John to make every multiple of 5¢ from 10¢ upwards. If we can find a denomination that will let us achieve 31¢, 32¢, 33¢, 34¢, 36¢, 37¢, 38¢, and 39¢ of postage, then we can achieve every higher denomination by adding 10¢ stamps.

To achieve a total of 31¢, we must add to one of the values we already can achieve. Starting at 0¢, 10¢, 15¢, 20¢, 25¢, or 30¢, we need to add 31¢, 21¢, 16¢, 11¢, 6¢, or 1¢, respectively. Any denomination of stamp that evenly divides any of these 6 integers will achieve 31 cents of postage. The following table shows all factors of the possible increases.

Starting value	0¢	10¢	15¢	20¢	25¢	30¢
Additional postage required	+31¢	+21¢	+16¢	+11¢	+6¢	+1¢
Possible denominations	1, 31	1, 3, 7, 21	1, 2, 4, 8, 16	1, 11	1, 2, 3, 6	1

Now we consider each of these possible denominations (in descending order) to see whether the denomination can achieve the other required postage values.

A 31¢ denomination won't help us achieve 32¢ in postage. We still need a penny, but the smallest denomination we have is 10¢.

A 21¢ denomination won't help us achieve 32¢ in postage. We can't use more than one 21¢ stamp. If we use one 21¢ stamp, we still need 11¢, which is not a multiple of 5¢.

A 16¢ denomination will achieve 32¢ in postage. But it won't help us achieve 33¢ in postage. If we use one 16¢ stamp, we will need 17¢ additional, which is not a multiple of 5¢. If we use two 16¢ stamps, we will need a penny.

An 11¢ denomination is more promising.  $32=10+11+1$ .  $33=11+11+1$ . But we cannot achieve 34¢ in postage. If we use exactly one 11¢ stamp, we still need 23¢. If we use exactly two 11¢ stamps, we still need 12¢. If we use three 11¢ stamps, we still need a penny. None of these are multiples of 5¢.

An 8¢ denomination is also promising.  $32=8+8+8$ .  $33=15+10+8$ .  $34=10+8+8+8$ .  $36=10+10+8+8$ . But we cannot achieve 37¢ in postage. If we use exactly one 8¢ stamp, we still need 29¢. If we use exactly two 8¢ stamps, we still need 21¢. If we use exactly three 8¢ stamps, we still need 13¢. If we use four 8¢ stamps, we still need a nickel.

Let's consider a 7¢ denomination.  $32=15+10+7$ . But we cannot achieve 33¢ in postage. If we use exactly one 7¢ stamp, we still need 26¢. If we use exactly two 7¢ stamps, we still need 19¢. If we use exactly three 7¢ stamps, we still need 12¢. If we use exactly four 7¢ stamps, we still need a nickel.

At last! The 6¢ denomination will yield all of the values we need.  $31=15+10+6$ .  $32=10+10+6+6$ .  $33=15+6+6+6$ .  $34=10+6+6+6+6$ .  $36=15+15+6$ .  $37=15+10+6+6$ .  $38=10+10+6+6+6$ .  $39=15+6+6+6+6$ . The answer is 6 cents.

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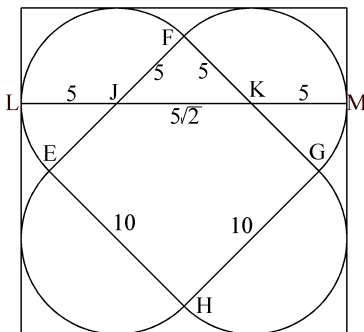
7. For a number to be divisible by 99, it must be divisible by both 9 and 11. One of the divisibility rules for 11 is that the positive difference between the sum of one set of alternate digits and the sum of the other set of alternate digits must be a multiple of 11. [For example, 92807 is divisible by 11 because  $9+8+7=24$ , and  $2+0=2$ . The difference is 22, which is a multiple of 11.] Given the 9-digit number,  $abb,aba,ba3$ , one set of alternate digits is  $a+b+b+b+3$ . The other set of alternate digits is  $b+a+a+a$ . The difference,  $2(b-a)+3$ , must be a multiple of 11. Pairs of digits  $(a, b)$  that satisfy this condition are:  $(0, 4)$ ,  $(1, 5)$ ,  $(2, 6)$ ,  $(3, 7)$ ,  $(4, 8)$ ,  $(5, 9)$ ,  $(7, 0)$ ,  $(8, 1)$ , and  $(9, 2)$ . Note the problem asks for the value of  $(b-a)$ , which must be either 4 or  $-7$ .

The well-known divisibility rule for 9 is that the sum of the digits must be a multiple of 9. Given the 9-digit number,  $abb,aba,ba3$ , the sum of the digits,  $4(a+b)+3$  must be a multiple of 9. The only pair of digits from the previous condition that satisfy this new condition are  $(1, 5)$ . So the 9-digit number is 155,151,513. Checking our work,  $155151513=1567187 \times 99$ .

The answer is 4.

8. Since the function is provided in factored form, the  $x$ -intercepts are:  $x=3$  and  $x=-2$ . At  $x=0$ , the  $y$ -intercept can be computed as  $y=(-3)^2 \times 2 = 9 \times 2 = 18$ . The base of this triangle is 5 units. The height is 18 units, so  $A = \frac{1}{2} \times b \times h = \frac{1}{2} \times 5 \times 18 = 45$ .

9. Begin with a drawing of the problem:



The diameter of the semicircles is given as 10 inches. Therefore, the radii of the semicircles are 5 inches and the side-length of square EFGH is 10 inches. Let points J and K be the midpoints of sides EF and FG, respectively. Line segment JK is extended to intersect one of the semicircles at point L. Line segment JK is extended to intersect another semicircle at point M. Points L and M are points of tangency of the semicircles to the outer square. JFK is a  $45^\circ$  right triangle. The length of the hypotenuse is  $5\sqrt{2}$ . The distance from L to M is the side-length of the outer square, and can be calculated as  $5 + 5\sqrt{2} + 5 \cong 5 + 7.07 + 5 = 17.07$ . The answer, rounded to the nearest tenth is 17.1 inches.

10. The distance from point A to point C is a space-diagonal of the cube. The length is given by  $s\sqrt{3} = 6\sqrt{3}$ . The distance from point X to point Y is the same as the length of the diagonal of a face. This length is given by  $s\sqrt{2} = 6\sqrt{2}$ . Because X and Y are given as midpoints of opposite sides, quadrilateral AXC Y is a rhombus. The area of a rhombus is given by  $A = \frac{d_1 \times d_2}{2} = \frac{6\sqrt{3} \times 6\sqrt{2}}{2} = \frac{36\sqrt{6}}{2} = 18\sqrt{6} \cong 44.09$ . To the nearest tenth, the answer is 44.1 square inches.