MATHCOUNTS

2001
■ National Competition ■
Sprint Round
Problems 1-30

The MATHCOUNTS Foundation claims copyright to the questions. Therefore, the questions are not included herein. As of this writing, the questions are available from the MATHCOUNTS Store as part of the 2001 Competition Set (MCC-01), at a price of $19.95 plus shipping and handling.

[Author's note: The 2001 Competition season marked the end of my second year of involvement with the MATHCOUNTS program. As I have often stated, I think the MATHCOUNTS program is an outstanding program for middle school students. Last year, I prepared solutions for the 2000 National Competition. I offered these solutions to MATHCOUNTS to be posted on their web site. MATHCOUNTS did not accept my offer. This year, I again offered to prepare solutions for the 2001 National Competition to be posted on the MATHCOUNTS web site. My offer, which has been outstanding for several months, has not been accepted. Once again, I am posting the solutions on my own web site.]

Permission is granted for use of these solutions in any nonprofit U. S. school participating in the MATHCOUNTS program. Permission to use these solutions for any commercial purpose or for any other purpose, including incorporation of these solutions into commercial "question databases" must be obtained from the author. Send permission requests to the author at mc@verser.org or to P. O. Box 1295, Loveland, Colorado 80539.

1. It is reasonable to assume that John's father's age is a 2-digit number. Since the reversed digits are John's age, his age is also a 2-digit number. There are only two combinations of two digits which total 5 – {1, 4} and {2, 3}. Since 41 – 14 = 27, we know John is 14 and his father is 41. The answer is 41.

2. Remember the formula, distance = rate × time. $d = r \times t$. To find Robbi's average speed, we need to know the total distance and the total time for all three legs of the ride. We can find the time to complete each leg of the ride by rewriting the formula as $t = \frac{d}{r}$. For the uphill leg, $t_{uphill} = \frac{3miles}{8mph} = \frac{3}{8}hour$. For the level leg, $t_{level} = \frac{7miles}{14mph} = \frac{1}{2}hour$. For the downhill leg, $t_{downhill} = \frac{12miles}{24mph} = \frac{1}{2}hour$. The total distance traveled is 3+7+12 = 22 miles. The total time spent is $\frac{3}{8} + \frac{1}{2} + \frac{1}{2} = \frac{11}{8}hours$. Rewrite the original formula once more as $r = \frac{d}{t}$. We plug in the total distance and the total time to find the average rate (speed):

$$r = \frac{d}{t} = \frac{22miles}{\frac{11}{8}hours} = 22 \div \frac{11}{8} = 22 \times \frac{8}{11} = 16mph$$

3. Each of the series is an arithmetic sequence. We could use the general formula for finding the sum of an arithmetic sequence: $\frac{first + last}{2} \times count$. However, in this case, we can solve this problem more quickly by using the special formulas for finding the sum of the first n even integers, $sum = n \times (n+1) = 20 \times 21 = 420$, and for the sum of the first m odd integers, $sum = m^2 = 15^2 = 225$. The difference between these two sums is $420 - 225 = 195$, which is the answer.

4. Examine the array. As long as the number of rows is even, the sum of each column will be equal. For example, the sums of the first two rows are 7, 7, and 7. Also note that the first two rows contain the first 6 integers. The sum of the first n integers is $sum = \frac{n \times (n+1)}{2}$. The first two rows contain the first 6 integers, whose sum is 21. Dividing this by 3, the sum of each column must be 7, which checks. Now let's do the real problem. The array contains the first 102 integers. Since 102 is divisible by 6, we know there are an even number of rows. The sum of these 102 integers is $sum = \frac{n \times (n+1)}{2} = \frac{102 \times 103}{2} = 5253$. Dividing 5253 by 3, the sum of each equal column is 1751.

5. Let's rewrite the problem statement as mathematical sentences:

$$a + b + c = 99$$

$$a + 6 = b - 6 = c \times 5$$

Now, let's rewrite the second equation to find a in terms of b. And again to find c in terms of b.

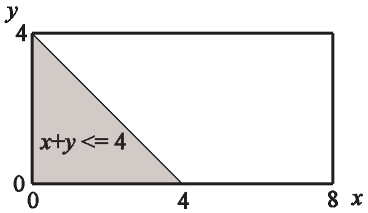
$$\begin{aligned} a + 6 &= b - 6 \\ a &= b - 12 \end{aligned}$$

$$\begin{aligned} b - 6 &= c \times 5 \\ \frac{b-6}{5} &= c \end{aligned}$$

Now, we can substitute for a and for c in the first equation, and then simplify:

$$\begin{aligned} a + b + c &= 99 \\ (b - 12) + b + \left(\frac{b-6}{5}\right) &= 99 \\ (5b - 60) + 5b + (b - 6) &= 495 \\ 11b - 66 &= 495 \\ 11b &= 561 \\ b &= 51 \end{aligned}$$

6. Here is a picture, showing the stated problem:



The rectangle represents the range from which the point (x, y) can be chosen. The area of this rectangle is 32.

The shaded triangle represents the range of interest. The area of this triangle is 8.

The probability that a point selected from the rectangle will fall within the shaded triangle is ¼.

7. There are a total of 10 vertices in the pentagonal prism. If we choose any vertex, we could draw a line segment to 9 other vertices. 3 of the 9 are adjacent. 6 of the 9 are non-adjacent From each vertex, we can draw 6 line segments. Since there are a total of 10 vertexes, we should be able to draw $10 \times 6 = 60$ line segments. However, we have drawn each line segment twice. Therefore, the total number of distinct diagonals is $60 \div 2 = 30$.

8. Label each bar as A, B, C, D. The key is that three bars are known to be gold. One bar is known to be counterfeit.

   Step 1. Balance A versus B. If they are unequal, we know A or B is the counterfeit, and that C and D are real gold. If they are equal, we know C or D is the counterfeit, and A and B are real gold. In either case, we have two known gold bars and we have two questionable bars.

   Step 2. Balance a known gold bar against one of the questionable bars. If they are unequal, we know for certain the questionable bar is the counterfeit. If they are equal, we know they are both real gold. This leaves only one questionable bar, which must be the counterfeit.

   It takes 2 weighings.

9. Let x be the total number of people at the party. The number of people at the party before Cedric arrived was $\frac{2}{3}x$. Cedric added 1. Plus 6 more people. The number of people at the party should now total $\frac{5}{6}x$. Write an equation, and solve:

$$\begin{aligned}
\frac{2}{3}x+1+6 &= \frac{5}{6}x \\
6\times\left(\frac{2}{3}x+1+6\right) &= 6\times\frac{5}{6}x \\
4x+6+36 &= 5x \\
4x+42 &= 5x \\
42 &= x
\end{aligned}$$

10. The first row has 10 seats. Row 2 has 11 seats. Row 29 has 38 seats. Row 30 has 39 seats. (There are always 9 more seats than the row number.)

    The first row will seat 5 students. The second row will seat 6 students. The third row will seat 6 students. The fourth row will seat 7 students. Etc. (If the number of seats is even, the number of students is $n \div 2$. If the number of seats is odd, the number of students is $(n+1) \div 2$.

    Let's make a table. The table will have columns showing the row number, the number of seats per row, and the number of students per row.

Row Number	Number of Seats	Number of Students
1	10	5
2	11	6
3	12	6
4	13	7
…	…	…
27	36	18
28	37	19
29	38	19
30	39	20

    Once you have this table, there are many ways to proceed. One way is to notice that number of students that can be seated in rows 1 and 30 is 5+20=25. The number of students that can be seated in rows 2 and 29 is 6+19=25. If we continue pairing rows like this, we will have 15 pairs of rows, with each pair of rows seating 25 students. The auditorium will seat 15×25=375 students, which is the answer.

11. Consider one side of the equilateral triangle. There are two squares, with a side common to the side of the triangle. (One square will lie completely outside the triangle. The other square will completely contain the triangle.) There is also one square, with a diagonal of the square common to the side of the triangle. Similarly, three squares could be drawn using each of the other sides of the triangle. The total number of squares is 3×3=9, which is the answer.

12. On Monday, 1 person already knew the secret (Jessica). She told 2 friends. 3 people know the secret on Monday.

    On Tuesday, 3 people already knew the secret. The 2 people who just learned the secret told 4 friends. 7 people know the secret by the end of the day on Tuesday.

    On Wednesday, 7 people already knew the secret. The 4 people who just learned the secret told 8 friends. 15 people know the secret by the end of the day on Wednesday.

    I see a pattern, here. By the end of the day on Thursday, 31 people will know the secret. On Friday, 63 people will know. On Saturday, 127 people will know. On Sunday, 255 people will know. On Monday, 511 people will know. On Tuesday, 1023 people will know. The answer is Tuesday.

13. There are seven sticks, all of differing lengths. Three sticks are drawn from a box. The total number of combinations of sticks that can be drawn is $\binom{7}{3} = \frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$. Since this is a manageable number, let's enumerate all possible combinations. Then we will see which choices make a triangle. (Three sticks will make a triangle if the sum of the lengths of the shortest two sticks is greater than the length of the longest stick.)

Shortest stick (S)	Middle stick (M)	Choices of longest stick (L)	Choices which make a triangle (S+M>L)
2	3	5, 7, 11, 13, 17	
2	5	7, 11, 13, 17	
2	7	11, 13, 17	
2	11	13, 17	
2	13	17	
3	5	7, 11, 13, 17	7
3	7	11, 13, 17	
3	11	13, 17	13
3	13	17	
5	7	11, 13, 17	11
5	11	13, 17	13
5	13	17	17
7	11	13, 17	13, 17
7	13	17	17
11	13	17	17

If we tally the number of combinations, there are indeed 35 enumerated, which is what we expect. If we tally the number of combinations which make a triangle, there are 9. So the probability of drawing three sticks which can make a triangle is $\frac{9}{35}$.

14. Since the area of the original square is 6, the side length of the original square is sqrt(6). Let l be the length, along the side which is folded up. (Point A will be $d = \sqrt{2} \times l$ units from its original position.) The black area is half of a square. Its area will be $\frac{l^2}{2}$. The white area is a square removed from a square. Its area will be $6 - l^2$. The problem says these areas are equal. Equate and solve:

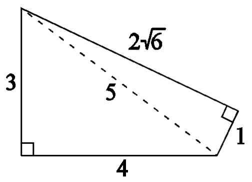
$$\begin{aligned}
\frac{l^2}{2} &= 6 - l^2 \\
2 \times \frac{l^2}{2} &= 2 \times (6 - l^2) \\
l^2 &= 12 - 2l^2 \\
3l^2 &= 12 \\
l^2 &= 4 \\
l &= 2
\end{aligned}$$

$$\begin{aligned}
d &= \sqrt{2} \times l \\
d &= \sqrt{2} \times 2 \\
d &= 2\sqrt{2}
\end{aligned}$$

15. If a and b are positive integers, the right hand side is a continued fraction representation of the left-hand side. To solve, invert the equation, drop the integer portion, and repeat.

$$\begin{aligned}
\frac{7}{22} &= \frac{1}{a + \frac{1}{b}} \\
\frac{22}{7} &= a + \frac{1}{b} \\
3 + \frac{1}{7} &= a + \frac{1}{b} && , a = 3 \\
\frac{1}{7} &= \frac{1}{b} \\
7 &= b && , \text{b} = 7 \\
0 &= 0
\end{aligned}$$

16. Add a line segment dividing the quadrilateral into two right triangles, with a shared hypotenuse.



Recognize the 3-4-5 triangle or use to Pythagorean theorem to find the length of the hypotenuse. Finally, use the Pythagorean theorem to find the length of the remaining side of the quadrilateral. $1^2 + x^2 = 5^2$. $x^2 = 5^2 - 1^2 = 25 - 1 = 24$. $x = \sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$. The area of the 3-4-5 triangle is $\frac{1}{2}bh = \frac{1}{2}4 \times 3 = 6$. The area of the other triangle is $\frac{1}{2}bh = \frac{1}{2}2\sqrt{6} \times 1 = \sqrt{6}$. So the total area of the quadrilateral is $6 + \sqrt{6}$.

17. [This problem contains an ambiguity. The problem permits the balls to be placed “in any combination” into two containers. The problem goes on to state that a ball is selected from a container. However, the problem does not state what happens if an attempt is made to draw a ball from an empty container. Based on the answer key, it appears the author of the question intended the balls to be placed in two containers—not “in any combination”, but with the restriction that each container must contain at least one ball. In my opinion, the most correct answer would be to state that “not enough information is provided”.]

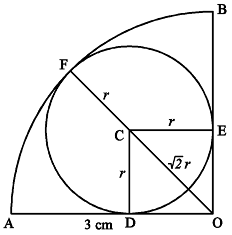
    You could enumerate all 36 combinations (since the two boxes are interchangeable, there are 18 unique combinations). However, intuition can guide you towards a likely solution.

    Consider if you put all of the red balls in container A and all of the green balls in container B. The chance of winning if container A is drawn is 0 in 5. The chance of winning if container B is chosen is 5 in 5. Overall, the chance of winning is ½. If we removed some green balls from container B, the chance of winning with that container will still be unity. If we put these balls in container A, the chance of winning with that container increases with each ball added. This intuition suggests that placing one green ball in container B and all the remaining balls in container A would be a good solution. In this case, if container A were chosen, the chance of winning is 4 in 9. If container B were chosen, then chance of winning is 1 in 1. Overall, the chance of winning is $\frac{1}{2}\times\frac{4}{9}+\frac{1}{2}\times\frac{1}{1}=\frac{1}{2}\times\left(\frac{4}{9}+1\right)=\frac{1}{2}\times\frac{13}{9}=\frac{13}{18}$.

    To be sure, we can make a table enumerating all possibilities:

Box A R+G	Box B R+G	X=p(winning if box A is chosen)	Y=p(winning if box B is chosen)	X+Y (approximate)	p(winning)= (X+Y)/2 (exact)
5+5	0+0	5/10 = 0.50	0/0 (Undefined)	Undefined	Undefined
5+4	0+1	4/9 = 0.44	1/1 = 1.00	1.44	13/18
5+3	0+2	3/8 = 0.38	2/2 = 1.00	1.38	11/16
5+2	0+3	2/7 = 0.29	3/3 = 1.00	1.29	9/14
5+1	0+4	1/6 = 0.17	4/4 = 1.00	1.17	7/12
5+0	0+5	0/5 = 0.00	5/5 = 1.00	1.00	1/2
4+5	1+0	5/9 = 0.55	0/1 = 0.00	0.55	5/18
4+4	1+1	4/8 = 0.50	1/2 = 0.50	1.00	1/2
4+3	1+2	3/7 = 0.43	2/3 = 0.67	1.10	23/42
4+2	1+3	2/6 = 0.33	3/4 = 0.75	1.08	13/24
4+1	1+4	1/5 = 0.20	4/5 = 0.80	1.00	1/2
4+0	1+5	0/4 = 0.00	5/6 = 0.83	0.83	5/12
3+5	2+0	5/8 = 0.63	0/2 = 0.00	0.63	5/16
3+4	2+1	4/7 = 0.57	1/3 = 0.33	0.90	19/42
3+3	2+2	3/6 = 0.50	2/4 = 0.50	1.00	1/2
3+2	2+3	2/5 = 0.40	3/5 = 0.60	1.00	1/2
3+1	2+4	1/4 = 0.25	4/6 = 0.67	0.92	11/24
3+0	2+5	0/3 = 0.00	5/7 = 0.71	0.71	5/14

18. Mark the center of the circle, C, and draw a radius to each of the three tangent points, which I label as D, E, and F.

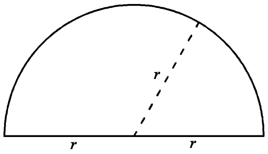


Because OAB is defined to be a quarter of a circle, line segments OA and OB are perpendicular to each other. Since the circle is tangent to the sector, radii CD and CE are perpendicular to OA and OB, respectively. Therefore, ODCE is a square. The length of the diagonal OC of square ODCE is $\sqrt{2}r$.

By symmetry, F is the midpoint of arc AB. Therefore, points O, C, and F are collinear. The length of |OF| is |OC|+|CF|, which is $\sqrt{2}r + r$. Since the length of |OF| is given as 3 cm, we can now solve for *r*:

$$\sqrt{2}r + r = 3$$
$$r\left(\sqrt{2}+1\right) = 3$$
$$r = \frac{3}{\sqrt{2}+1}$$
$$r = \frac{3}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$
$$r = \frac{3\sqrt{2}-3}{2-1} = 3\sqrt{2}-3$$

19. Sketch the semicircular garden.



The area of this figure is half the area of a circle, $A = \frac{1}{2}\pi r^2$.

The perimeter of this figure is half the perimeter of a circle, plus two radii:

$$P = \frac{1}{2}2\pi r + 2r = \pi r + 2r = (2+\pi)r$$

Now, we simply set the perimeter equal to twice the area, and solve for *r*.

$$(2+\pi)r = 2\times\left(\frac{1}{2}\pi r^2\right) = \pi r^2$$
$$2+\pi = \pi r$$
$$r = \frac{2+\pi}{\pi} = \frac{2}{\pi}+1$$

20. The last equation is in terms of *o*, *n*, and *s*. Rewrite the first three equations to express *o*, *n*, and *s* in terms of *u*, *c*, and *t*. Then substitute in the last equation and solve. The *u* and *c* terms cancel each other out.

$$\begin{aligned} c+o&=u & o&=u-c \\ u+n&=t & n&=t-u \\ t+c&=s & s&=t+c \end{aligned}$$

$$\begin{aligned} o+n+s&=12 \\ (u-c)+(t-u)+(t+c)&=12 \\ u-c+t-u+t+c&=12 \\ u-u-c+c+t+t&=12 \\ 2t&=12 \\ t&=6 \end{aligned}$$

21. Multiply numerator and denominator by (x-1), and by (x+1). This will eliminate the compound fractions. Then expand the quadratic equations and combine like terms.

$$\frac{\frac{x+1}{x-1}-\frac{x-1}{x+1}}{\frac{x+1}{x-1}+\frac{x-1}{x+1}}\times\frac{x-1}{x-1}$$

$$\frac{x+1-\frac{(x-1)^2}{x+1}}{x+1+\frac{(x-1)^2}{x+1}}\times\frac{x+1}{x+1}$$

$$\frac{(x+1)^2-(x-1)^2}{(x+1)^2+(x-1)^2}$$

$$\frac{(x^2+2x+1)-(x^2-2x+1)}{(x^2+2x+1)+(x^2-2x+1)}$$

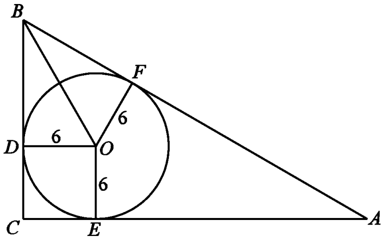
$$\frac{2x+2x}{2x^2+2}$$

$$\frac{2x}{x^2+1}$$

22. I believe this problem requires you to know some of characteristics of a 30-60-90 triangle.

    The first method uses the somewhat obscure formula $r = \frac{2A}{P}$. I.e., the radius of a circle inscribed in a triangle is equal to twice the area of the triangle divided by the perimeter of the triangle. The radius is known, and the area and perimeter are fairly simple functions of the length of side AB. Solving for the length of side AB is straightforward.

    The second method is probably a little simpler for this problem.



The inscribed circle is tangent to each side of the triangle. The points of tangency are marked *D*, *E*, and *F*, and a radius is drawn to each point of tangency. Each radius makes a right angle with the corresponding side of the triangle.

An additional line segment is drawn from *O* to *B*. Triangles *ODB* and *OFB* are congruent 30-60-90 triangles. By the properties of a 30-60-90 triangle, $|DB| = |OD| \times \sqrt{3} = 6\sqrt{3}$.

Since *OECD* is a square, the length of $\overline{CD}$ is 6. The length of $\overline{CB}$ is $6 + 6\sqrt{3}$. The length of the hypotenuse, $\overline{AB}$, is twice the length of $\overline{CB}$, or $12 + 12\sqrt{3}$.

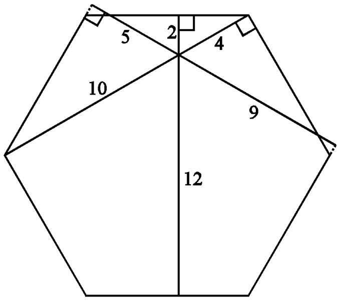
23. For this problem, I'll make a table containing *n*, *n*-3, |*n*|, and |*n*-3|. Then test which values meet the condition of the problem

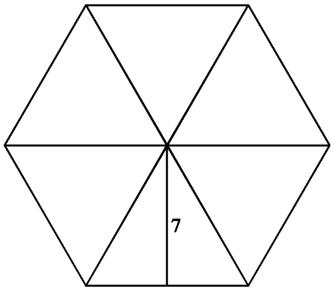
n	–7	–6	–5	–4	–3	–2	–1	0	+1	+2	+3
n-3	–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0
|*n*|	+7	+6	+5	+4	+3	+2	+1	0	+1	+2	+3
|*n*-3|	+10	+9	+8	+7	+6	+5	+4	+3	+2	+1	0
|*n*|<|*n*-3|<9	F	F	T	T	T	T	T	T	T	F	F
n, where condition is true			–5	–4	–3	–2	–1	0	+1		

The answer is the sum of the values, *n*, where the condition is true.
$-5 - 4 - 3 - 2 - 1 + 0 + 1 = -14$.

24. In the top row, alone, there are 10 rectangles. In the bottom row, alone, there are 10 more rectangles. There are 3 more rectangles that are two rows high. A total of 23 rows are present.

25. In a regular hexagon, there are three pairs of opposite sides, which are parallel and equidistance apart. The average length of the specified perpendiculars is 7. As it happens, there are three pairs of perpendiculars which total 14, {2+12, 4+10, and 5+9}.



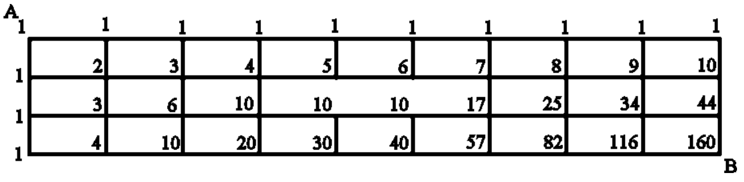


A regular hexagon can be drawn as six congruent equilateral triangles with a common vertex. The information supplied indicates the altitude of these triangles is 7. For an equilateral triangle, $a=\sqrt{3}\times\frac{s}{2}$. Rewriting, $s=\frac{2a}{\sqrt{3}}=\frac{2\times 7}{\sqrt{3}}=\frac{14}{\sqrt{3}}=\frac{14}{3}\sqrt{3}$.

26. Since this problem involves base-6, the first several powers of 6 may be useful. They are 1, 6, 36, and 216. Aha! This problem will be simple if we convert the multiplicand and multiplier to base 6, first. $217_{10}=216+1=1001_6$ and $45_{10}=36+6+1+1+1=113_6$. Now we perform the multiplication in base 6: $1001_6\times 113_6=113113_6$. The units digit is 3.

We can also solve this problem using modular arithmetic. 217 modulo 6 = 1. 45 modulo 6 = 3. $(217\times 45)\bmod 6=(217\bmod 6)\times(45\bmod 6)=1\times 3=3$. The base-6 units digit of the product is 3.

27. My plan is to label each node (intersection) of the map with the number of ways of arriving at that node. The problem says we start at point A. Begin by labeling point A with "1", since there is just one way to get there. For each of the remaining nodes, we can arrive at the node from the left or from above. Therefore, the number of ways of getting to a node equal the sum of the number of ways of getting to the node above plus the number of ways of getting to the node to the left. In several cases, there is no path from the left or from above. Following is my labeled diagram, which I completed by working from the upper left to the lower right. The completed diagram shows the total number of ways of arriving at point B is 160, which is the answer. [Note the resemblance of the upper left corner of our graph with Pascal's Triangle.]



1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	10	10	17	25	34	44
1	4	10	20	30	40	57	82	116	160

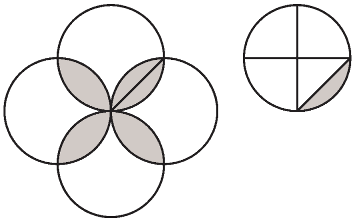
28. Here, we manipulate some equations symbolically, then substitute the known values specified in the problem:

$$\begin{aligned}
(a+b)(a^2+b^2) &= a^3+a^2b+ab^2+b^3 \\
3(a+b)(a^2+b^2) &= 3a^3+3a^2b+3ab^2+3b^3 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
3(a+b)(a^2+b^2)-(a+b)^3 &= 2a^3+2b^3 \\
3(1)(2)-(1)^3 &= 2(a^3+b^3) \\
5 &= 2(a^3+b^3) \\
a^3+b^3 &= \frac{5}{2}
\end{aligned}$$

Alternatively, we can rewrite the first equation as $b = 1-a$. Substituting for b in the second equation, and simplifying, we have the quadratic equation, $2a^2-2a-1=0$. Using the quadratic formula, we can find $a = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$, and then $b = \frac{1}{2} \mp \frac{\sqrt{3}}{2}$. Compute:

$$\begin{aligned}
a^3 &= \frac{5}{4} \pm \frac{3\sqrt{3}}{4} \\
b^3 &= \frac{5}{4} \mp \frac{3\sqrt{3}}{4} \\
a^3+b^3 &= \frac{5}{2}
\end{aligned}$$

29. The problem asks us to find the total area of the shaded sections. In the circle to the right the shaded section is 1/8 of the total shaded area on the left.



The shaded section in the circle to the right is the area of a quarter circle minus the area of the triangle. The area of the quarter circle is $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi 4^2 = 4\pi$. The area of the triangle is $\frac{b \times h}{2} = \frac{4 \times 4}{2} = 8$. The shaded area is $4\pi - 8$. The total area is 8 times this, or $32\pi - 64$.

30. First, I'll rewrite the word problems in number sentences:

$$C_B = 1.6 \times C_F$$
$$V_B = 0.75 \times V_G$$
$$C_G = 0.75 \times C_B$$
$$V_G = \frac{4}{3} \times V_F$$

The problem tells us that Fresh costs $1.00 per unit volume. Let's assume a 3 ounce package of Fresh costs $3.00. $C_F = \$3.00$, $V_F = 3oz$.

Now, we can solve the equations to find the cost and volume of Glow:

$$C_B = 1.6 \times C_F = 1.6 \times 3.00 = \$4.80$$
$$V_G = \frac{4}{3} \times V_F = \frac{4}{3} \times 3 = 4oz$$
$$C_G = 0.75 \times C_B = 0.75 \times 4.80 = \$3.60$$

Finally, the problems asks for the number of cents per unit volume for Glow, which is the cost divided by the volume:

$$\frac{C_G}{V_G} = \frac{3.60}{4} = \$.90 = 90 \text{ cents}$$

MATHCOUNTS

2001
■ National Competition ■
Target Round
Problems 1-8

The MATHCOUNTS Foundation claims copyright to the questions. Therefore, the questions are not included herein. As of this writing, the questions are available from the MATHCOUNTS Store as part of the 2001 Competition Set (MCC-01), at a price of $19.95 plus shipping and handling.

[Author's note: The 2001 Competition season marked the end of my second year of involvement with the MATHCOUNTS program. As I have often stated, I think the MATHCOUNTS program is an outstanding program for middle school students. Last year, I prepared solutions for the 2000 National Competition. I offered these solutions to MATHCOUNTS to be posted on their web site. MATHCOUNTS did not accept my offer. This year, I again offered to prepare solutions for the 2001 National Competition to be posted on the MATHCOUNTS web site. My offer, which has been outstanding for several months, has not been accepted. Once again, I am posting the solutions on my own web site.]

Permission is granted for use of these solutions in any nonprofit U. S. school participating in the MATHCOUNTS program. Permission to use these solutions for any commercial purpose or for any other purpose, including incorporation of these solutions into commercial "question databases" must be obtained from the author. Send permission requests to the author at mc@verser.org or to P. O. Box 1295, Loveland, Colorado 80539.

1. The problem shows four points arranged in a unit square. If we choose any 2 points, there are two distinct unit circles that pass through the two points. (The center of one circle is on one side of the line between the points. The center of the other circle is on the other side of the line between the points.) Since the number of combinations of 4 things taken 2 at a time is 6, and there are two circles per pair, there are a maximum of 12 circles. The centers of four of the circles are outside the square. The centers of four of the circles are inside the square. And the centers of four of the circles correspond to corners of the square. Since none of the 12 circles are repeated, the answer is 12.

2. Let I represent Otto's initial investment in internet stocks. Using just the second sentence, we know the total value of Otto's portfolio at the end of the year is $1.10\times I+9000$. Using just the third sentence, we know the total value of Otto's portfolio at the end of the year is $1.06\times(I+10000)$. We can set these equal and solve:

$$\begin{aligned}1.10\times I+9000&=1.06\times(I+10000)\\1.10\times I+9000&=1.06\times I+10600\\0.04\times I&=1600\\I&=40000\\1.1\times I&=44000\end{aligned}$$

   At the end of the year, the value of the internet stocks is $44,000.

3. The ratio of the surface area of sphere A to the surface area of sphere B is 1.96:1. The ratio of the radius of sphere A to the radius of sphere B is $\sqrt{1.96}:\sqrt{1}=1.4:1$. The ratio of the volume of sphere A to the volume of sphere B is $1.4^3:1^3=2.744:1$. Therefore, the volume of sphere A is 174.4% more than the volume of sphere B. Rounded to the nearest whole number, the answer is 174%.

4. We are given f(1) and f(2), and a formula for finding f(n), given f(n-1) and f(n-2). We solve for f(3), f(4), and f(5):x

$$\begin{aligned}&f(1) = -1\\ &f(2) = 3\\ &f(3) = 3f(1) - 2f(2) = -3 - 6 = -9\\ &f(4) = 3f(2) - 2f(3) = 3\times 3 - 2\times(-9) = 9 + 18 = 27\\ &f(5) = 3f(3) - 2f(4) = 3\times(-9) - 2\times 27 = -27 - 54 = -81\end{aligned}$$

The answer is –81.

5. The number of permutations of 4 things is 4! = 24. This is sufficiently few so that we can list all of the permutations. Assuming the CDs properly would be in the order A B C D, we mark the permutations where exactly two CDs are in the wrong case:

A B C D	$\underline{\text{B A C D}}$	C A B D	D A B C
$\underline{\text{A B D C}}$	B A D C	C A D B	D A C B
$\underline{\text{A C B D}}$	B C A D	$\underline{\text{C B A D}}$	D B A C
A C D B	B C D A	C B D A	$\underline{\text{D B C A}}$
A D B C	B D A C	C D A B	D C A B
$\underline{\text{A D C B}}$	B D C A	C D B A	D C B A

There probability that exactly two CDs are in the wrong case is $\frac{6}{24} = \frac{1}{4}$.

We could also have solved this by finding the number of combinations of 4 things taken 2 at a time (6), and divided that by the total number of permutations (24).

6. I symbolically list each of the six distinct three-digit numbers, and prepare to sum them:

$$\begin{array}{r} abc\\ acb\\ bac\\ bca\\ cab\\ \underline{+cba}\end{array}$$

The sum of each place position is $2a+2b+2c$. Therefore, the actual sum is equal to

$$200a + 200b + 200c + 20a + 20b + 20c + 2a + 2b + 2c = 222a + 222b + 222c = 222(a + b + c).$$

The greatest positive integer divisor of this sum, regardless of the choices of a, b, and c, is 222.

7. Triangle BFG is similar to triangle BHC. The length of BF is in the ratio 1:3 to the length of BH, so the areas of the triangles are in the ratio 1:9. The area of triangle BHC is $\frac{1}{2}b\times h = \frac{1}{2}\times 18\times 24 = 216$. Therefore, the area of triangle BFG is 216/9 = 24. The area of quadrilateral CGFH is 216-24 = 192. Since the figure is symmetric about line CH, the area of the pentagon CDEFG is twice the area of quadrilateral CGFH, or 384.

8. Since x and y are single-digit prime numbers, they must be 2, 3, 5, or 7. There are only 12 possible combinations, so we can enumerate all of them:

x	y	$10x+y$	$(x)(y)(10x+y)$
7	5	75 (not prime)	
7	3	73	1533
7	2	72 (not prime)	
5	7	57 (not prime)	
5	3	53	795
5	2	52 (not prime)	
3	7	37	777
3	5	35 (not prime)	
3	2	32 (not prime)	
2	7	27 (not prime)	
2	5	25 (not prime)	
2	3	23	138

The largest 3-digit product is 795, which is the answer.

MATHCOUNTS

2001
■ National Competition ■
Team Round
Problems 1-10

The MATHCOUNTS Foundation claims copyright to the questions. Therefore, the questions are not included herein. As of this writing, the questions are available from the MATHCOUNTS Store as part of the 2001 Competition Set (MCC-01), at a price of $19.95 plus shipping and handling.

[Author's note: The 2001 Competition season marked the end of my second year of involvement with the MATHCOUNTS program. As I have often stated, I think the MATHCOUNTS program is an outstanding program for middle school students. Last year, I prepared solutions for the 2000 National Competition. I offered these solutions to MATHCOUNTS to be posted on their web site. MATHCOUNTS did not accept my offer. This year, I again offered to prepare solutions for the 2001 National Competition to be posted on the MATHCOUNTS web site. My offer, which has been outstanding for several months, has not been accepted. Once again, I am posting the solutions on my own web site.]

1. The goal is to keep the volume of the jar constant. Since volume is a three-dimensional measurement, the volume is proportional to the height times the square of the radius of the jar. If we assume the old volume and the old radius were each unity, then the product of the new height times the square of the new radius must remain equal to 1:

$$\begin{aligned} h_{old} \times r_{old}^2 &= 1 \\ 1 \times 1^2 &= 1 \\ h \times r^2 &= 1 \\ 1.4 \times r^2 &= 1 \\ r &\cong 0.8451 \end{aligned}$$

The radius has decreased by about 15.49% from its original value. The answer is 15.5%.

2. Let a, b, and c be the smallest number, the middle number, and the largest number. I write number sentences for each statement in the problem:

$$\begin{aligned} a+b+c&=165\\ 7a&=n\\ c-9&=n\\ b+9&=n \end{aligned}$$

Restate a, b, and c in terms of n, and substitute into the first equation:

$$\begin{aligned} a&=\frac{n}{7}\\ c&=n+9\\ b&=n-9\\ a+b+c&=165\\ \left(\frac{n}{7}\right)+(n-9)+(n+9)&=165\\ \left(\frac{n}{7}\right)+2n&=165\\ n+14n&=165\times 7\\ 15n&=1155\\ n&=77\\ a&=\frac{n}{7}=\frac{77}{7}=11\\ b&=n-9=77-9=68\\ c&=n+9=77+9=86\\ a+b+c&=11+68+86=165 \end{aligned}$$

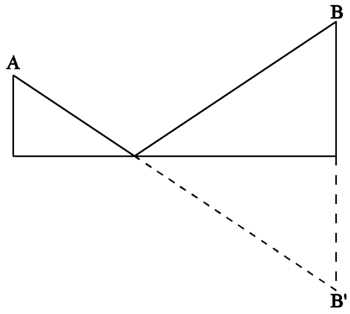
Finally, the product of a, b, and c is $11\times 68\times 86=64328$.

3. This problem can be solved by algebra, but it may be quicker to use a Guess and Check table. Let's guess the number of free hours. Then compute the price per hour for Wells and Ted and compute the price per hour for Vino. When they match, we should have our answer.

Free Hours (f)	Wells &Ted's Paid Hours (105 – 2f)	Wells & Ted's Price per Hour ($10/pd)	Vino's PaidHours (105 – f)	Vino's Price per Hour ($26/pd)
10	85	~$0.1176	95	~$0.2737
20	65	~$0.1538	85	~$0.3058
30	45	~$0.2222	75	~$0.3467
40	25	$0.40	65	$0.40
50	5	$2.00	55	$0.4727

The rates for Wells and Ted match the rate for Vino when there are 40 free hours. The rate is 40 cents per extra hour.

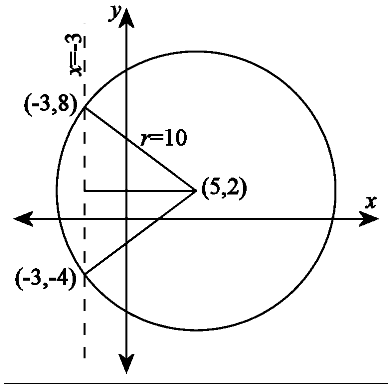
4. As can be seen in the problem diagram, all runners must run 1200 meters in the East-West direction. All runners must also run 800 meters (300 meters + 500 meters) in the North-South direction. Using the Pythagorean theorem, the shortest possible distance is $\sqrt{1200^2 + 800^2} \cong 1442.221$. This can be visualized by replacing the wall with a mirror. The shortest distance from A to B, which touches the wall, is the same as the distance from A to the reflection of B in the mirror. The answer is 1442 meters.



5. If Bob rolls 2, 3, or 5, he will eat unsweetened cereal. If he rolls 4 or 6 he will eat sweetened cereal. If he rolls a 1, he'll just keep rolling again and again until he rolls a 2 through 6. On any given day, the chances he'll eat unsweetened cereal are 3/5, and the chances he'll eat sweetened cereal are 2/5. Loosely speaking, in a year of 365 days, Bob is most likely to eat unsweetened cereal about 3/5 of the days, or about 219 days. He is most likely to eat sweetened cereal about 2/5 of the days, or about 146 days. So he is most likely to eat unsweetened cereal about 73 days more than he'll eat sweetened cereal.

6. The diagonal of the large cube is $\sqrt{10^2 + 10^2 + 10^2} = 10\sqrt{3}$. The diagonal of the small cube is $4\sqrt{3}$. Since the small cube is centered within the large cube, these diagonals are collinear. The distance from the corner of one cube to the nearest corner of the other cube is half the difference in the length of each cube's diagonals. The answer is $3\sqrt{3}$.

7. Sketch a Cartesian coordinate system. Sketch the line x=–3. Also, the set of points 10 units from the point (5,2) is a circle centered about the point.

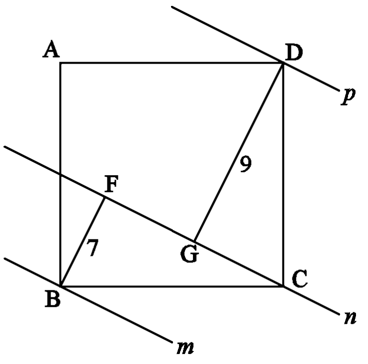


The problem requires you to find the *y*-coordinates which are on the circle and on the line *x*=–3. If you notice that the *x*-distance is 8 units (5 – (-3) = 8), and if you notice the diagonal of the triangle is 10, you should recognize you have 3-4-5 triangles. The *y*-distance is therefore 6 units. (You can also use the Pythagorean theorem or the distance formula, which are equivalent.) The *y*-values must be $2 \pm 6 = 8$ or –4. Finally, we find the product of these *y*-coordinates = –32.

8. The probability of rolling a 1 or a 6 is 1/3. The probability of tossing a heads is 1/2. The probability of both events (of winning on a given turn) is $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$. So the probability of not winning on a given turn is $1-\frac{1}{6}=\frac{5}{6}$. The probability of not winning on two consecutive turns is $\frac{5}{6}\times\frac{5}{6}$. And the probability of not winning on three consecutive turns is $\left(\frac{5}{6}\right)^3=\frac{125}{216}$. So the probability of winning on or before the third turn is $1-\frac{125}{216}=\frac{91}{216}$. This is the answer.

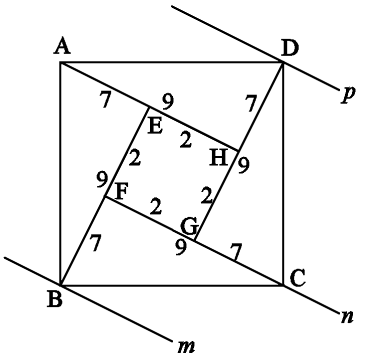
9. Rather than solving this symbolically, it may be easier to just plug in a number for u. Let's try u=1. Then we have the three side lengths equal to 1, $\sqrt{3}$, and 2. This should be recognized as a 30-60-90 triangle. In this case, the largest angle is 90 degrees, so that should be the answer. However, we now have a theory that's easy to prove. Sum the squares of the smaller two sides and compare with the square of the larger side. The squares of the smaller sides are $2u$-1 and $2u$+1. The sum is $4u$. The square of the largest side is also $4u$. The sides fit the Pythagorean theorem, regardless of u, so the largest angle must be 90 degrees.

10. Draw a perpendicular line segment from B to line *n*. Label the point of intersection as F, and label the length of this line segment as 7. Draw a perpendicular line segment from D to line *n*. Label the point of intersection as G and label the length of this line segment as 9.



BFC is a right angle. CGD is a right angle. Angles BCF and GCD are complementary. This makes triangle BFC similar to triangle CGD. Since the hypotenuses of these triangles are sides of the square, the triangles are congruent.

Draw a line segment from A perpendicular to DG. Label the point of intersection as H. Extend line segment BF to intersect AH at point E.



Our diagram is now complete. We have four congruent triangles, BFC, CGD, DHA, and AEB. Each of these triangles is a right triangle with legs 7 and 9. The area of each of these triangles is $\frac{1}{2}b\times h=\frac{1}{2}7\times 9=31.5$ units. The total area of the triangles is 126 units. In the center is square EFGH. The side length of this square is 2 and its area is 4. The total area of square ABCD is 126+4=130 square units.