Phil. 2440
Chapter 12: Metalogic

To Discuss:
- Basic concepts of metalogic
- Metalogic for the propositional calculus
- Metalogic for the predicate calculus

Basic concepts of metalogic

Logic vs. meta-logic

Some of the defects of ordinary language:
- Systematic ambiguities.
  - Ex.: “All politicians are not honest.”
- Metaphysics
- Sentences with misleading grammatical structures.
  - Ex.: “The average man has 2.3 children.”

The idea of a ‘logically perfect language’:
- No systematic ambiguities.
- No meaningless sentences
- Grammatical structure reflects logical structure
- Logical properties can be read off the syntactic structure

Formal systems
- Formation rules
- Axioms
- Transformation rules

Other concepts:
- ‘Arguments’
- ‘Proofs’
- ‘Theorems’
- ‘Interpretation’ of a system
- ‘Model’ of a set of sentences

Desirable properties of formal systems

Completeness:
- Every sentence that is true in all intended interpretations is a theorem.

Consistency:
- No sentence of the form $p \ \& \sim p$ is a theorem.

Soundness:
- No false sentences (in the intended interpretation) are theorems.

The consistency of the propositional calculus

Axioms:
- Law of excluded middle. $p \vee \sim p$
- Law of non-contradiction. $\sim(p \ \& \sim p)$

Interpretations in propositional logic:

Assign truth-values to atomic sentences

Consistency proof:

Lemma: In the propositional calculus, every theorem is a tautology, i.e., a proposition that is true in every intended interpretation.

- A) All the axioms of the propositional calculus are tautologies.
- B) Each of the transformation rules of the propositional calculus preserves tautologousness. That is, if you start from tautologies, they will enable you to derive only other tautologies.
- C) Therefore, all the theorems of the propositional calculus are tautologies, since they are derived from the axioms using the transformation rules.

1. In the propositional calculus, every theorem is a tautology.
2. Some propositions are not tautologies.
3. Therefore, some propositions are not theorems of the propositional calculus. (from 1,2)
4. If the propositional calculus is inconsistent, then every proposition is a theorem of it.
5. Therefore, the propositional calculus is consistent. (from 3,4)

Completeness of the propositional calculus

Conjunctive normal form:

Basic idea: One or more conjuncts. Each conjunct is a disjunction of one or more sentences. Each disjunct is an atomic sentence of the negation of an atomic sentence.

More precisely:

- a. There are no →'s or ↔'s.
- b. All ~'s apply to atomic sentences.
- c. All ∨'s apply to atomic sentences or negated atomic sentences.

Examples: which of these are in conjunctive normal form?

(A ∨ ~A)
A & (B ∨ C)
(B ∨ C) & (~C ∨ ~A)
(A ∨ B) & (B ∨ C ∨ A) & (C ∨ A)
A → (B ∨ C)
~(A & ~A)
(A & B) ∨ ~C

How to transform a sentence into conjunctive normal form:

Apply Impl. & Equiv.
Apply DeM
Apply Dist.

Example 1: Transform "A → ~(B ∨ C)" into conjunctive normal form.

1. A → ~(B ∨ C)
2. ~A ∨ ~(B ∨ C) 1 impl
3. ~A ∨ (~B ∨ ~C) 2 DeM

Example 2: Transform "~(A ↔ B) ∨ C" into conjunctive normal form.

1. ~(A ↔ B) ∨ C
2. ~[(A & B) ∨ (~A & ~B)] ∨ C 1 equiv
3. [~(A & B) & ~(~A & ~B)] ∨ C 2 DeM

4.	[(~A ∨ ~B) & ~(~A & ~B)] ∨ C	3 DeM
5.	[(~A ∨ ~B) & (~~A ∨ ~~B)] ∨ C	4 DeM
6.	[(~A ∨ ~B) & (A ∨ B)] ∨ C	5 DN (twice)
7.	[(~A ∨ ~B) ∨ C] & [(A ∨ B) ∨ C]	6 Dist
8.	(~A ∨ ~B ∨ C) & (A ∨ B ∨ C)	rewriting 7

Proving a sentence in conjunctive normal form:

- Each conjunct must be tautologous
- So each disjunction must be tautologous
- So each disjunction must contain an atomic sentence & its negation

Example 3: To prove: A → (B → A)

1.	A ∨ ~A	axiom
2.	~A ∨ A	1 comm
3.	(~A ∨ A) ∨ ~B	2 add
4.	~A ∨ (A ∨ ~B)	3 assoc
5.	~A ∨ (~B ∨ A)	4 comm
6.	A → (~B ∨ A)	5 impl
7.	A → (B → A)	6 impl

Metalogic for predicate calculus

Interpretations:

- Domain of discourse
- An object assigned to each constant
- A set of objects assigned to each predicate (its extension)
  - For relational predicates: assign a set of ordered pairs (triples, etc.)

Models:

A model for a set of sentences = an interpretation that makes all the sentences true.

Example:

(∃x)(∃y) Rxy
(∃x)(y) ~Rxy
(x)(∃y) Ryx

A model:

Domain of discourse = all natural numbers. R = the “successor” relation.

Desirable properties of predicate logic:

- Consistency
- Completeness
- Soundness

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Chapter 13: Gödel's Theorem

To Discuss:

Gödel's Theorem
Gödel's Second Theorem

What Is Gödel's Theorem?

Originally a response to *Principia Mathematica*
Applies to any other formal system of arithmetic
Gödel's Theorem:

> Any formal system capable of representing arithmetic on the natural numbers is either inconsistent or incomplete.

What G's Theorem does not say:

- Every formal system is inconsistent or incomplete.
- Anything about "knowledge".
- There are truths of arithmetic that cannot be *proven* in the standard English sense.
- There are truths of arithmetic that cannot be proven in any formal system.
- Anything about limits to human reason, the human mind, etc.

Outline of the Proof Procedure

Background: The liar paradox

(S) Statement S is false.

A Gödel sentence:

(G) Statement G cannot be proven in *Principia Mathematica*.

More precisely: The Gödel sentence for PM is a sentence of arithmetic that must be true if and only if it is not possible to derive that sentence using the rules of PM.

A little more detail:

Step 1: Number the sentences (and arguments) of PM.
Step 2: Show that the Gödel # of any sentence will have a specific arithmetical property, if and only if the sentence can be proven in PM.
Step 3: Formulate a sentence of PM that says that its own Gödel # does not have that property.

Step 1: Gödel Numbering

Goal of this section: To assign numbers to sentences & arguments in a formal system. I.e., to map sentences/arguments one-one onto a subset of the natural #s.

Numbering the basic symbols:

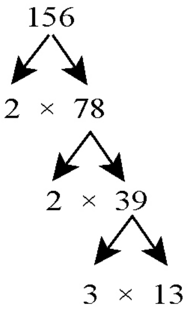
Symbol	Gödel number
(	1
)	2
∃	3
∨	4
~	5

0	6
s	7
=	8
+	9
×	10

Symbol	Gödel number	Symbol	Gödel number
x	11	A	12
y	13	B	14
z	15	C	16
⋮	⋮	⋮	⋮

Prime factorization:

All numbers have a unique prime factorization



Numbering the sentences:

Take the *n*th symbol in the sentence.

Find its Gödel #. Suppose it is *a*.

Take the *n*th prime number raised to the *a* power.

Example 1:

Find the Gödel number for the sentence, "0 = 0"

Answer:

String of symbols:	0	=	0
	↓	↓	↓
Gödel numbers for the symbols:	6	8	6
Series of prime numbers:	2	3	5
Gödel # for the string:	2^6	3^8	5^6

Answer: $2^6 \times 3^8 \times 5^6 = 6{,}561{,}000{,}000$

Example 2:

Find the sentence, if any, corresponding to the number 11,049,048,188,640.

Answer:

The prime factorization is $2^5 \times 3^2 \times 5^1 \times 7^8 \times 11^3$.

Prime factorization:	2^5 ×	3^2 ×	5^1 ×	7^8 ×	11^3
	↓	↓	↓	↓	↓
Gödel #s of symbols in the string:	5	2	1	8	3
The string:	~	)	(	=	∃

Numbering Arguments:

Take the *n*th sentence in the argument.
Find its Gödel #. Suppose it is *a*.
Take the *n*th prime number raised to the *a* power.

Example 3:

Find the Gödel # for the argument:

(x) x+0 = x
0+0 = 0
(∃x) x+x = x

Answer:

Sentences in the proof	Gödel numbers	For short
(x) x+0 = x	$2^1 \times 3^{12} \times 5^2 \times 7^{12} \times 11^9 \times 13^6 \times 17^8 \times 19^{12}$	*a*
0+0 = 0	$2^6 \times 3^9 \times 5^6 \times 7^8 \times 11^6$	*b*
(∃x) x+x = x	$2^1 \times 3^3 \times 5^{12} \times 7^2 \times 11^{12} \times 13^9 \times 17^{12} \times 19^8 \times 23^{12}$	*c*

Answer:

$2^a \times 3^b \times 5^c =$

$$2^{(2^1\cdot 3^{12}\cdot 5^2\cdot 7^{12}\cdot 11^9\cdot 13^6\cdot 17^8\cdot 19^{12})}\cdot 3^{(2^6\cdot 3^9\cdot 5^6\cdot 7^8\cdot 11^6)}\cdot 5^{(2^1\cdot 3^3\cdot 5^{12}\cdot 7^2\cdot 11^{12}\cdot 13^9\cdot 17^{12}\cdot 19^8\cdot 23^{12})}$$

Step 2: Correlating syntactic properties of sentences with arithmetical properties of Gödel numbers

Goal of this section: To show that there is an arithmetical property possessed by the Gödel numbers of valid arguments in PM.

Each syntactic property (of a sentence) corresponds to an arithmetical property (of a Gödel #).
Examples:

Syntactic remark about sentence	Arithmetical statement about Gödel #
S begins with "(".	The Gödel number of S is divisible by 2 but not by 4.
S contains "~~" somewhere.	There are consecutive prime numbers *n* and *m*, such that the Gödel number of S is divisible by ($n^5 \times m^5$) but not by n^6 or m^6.
⋮	⋮

Syntactic properties of *arguments* also correspond to arithmetical properties of Gödel #s.
Examples:

Argument A has 3 steps: The Gödel # of A is divisible by 2, 3, and 5, but not by any prime # greater than 5.

The operation of removing a double negation from the front of a sentence:

1. ~~0 = 0 $2^5 \times 3^5 \times 5^6 \times 7^8 \times 11^6$
2. 0 = 0 $2^6 \times 3^8 \times 5^6$

Getting an arithmetical property of the Gödel #s of theorems:

For any syntactic operation, there is a corresponding mathematical (arithmetic) operation.
So there is a mathematical relationship corresponding to each rule of the formal system.
So there is a mathematical relationship corresponding to *following the rules of the system*.
So there is a mathematical property of a sequence that follows the rules of the system.
So there is a mathematical property that the Gödel # of an argument has, if that argument is a proof in the system.
So there is a mathematical property that the Gödel # of a *sentence* has, if *there exists* a proof of that sentence in the system. Suppose this property is represented by $\phi(y)$.
So the formula

$\phi(y)$

is true (in the intended interpretation of the formal system) if and only if y is a theorem of the system.

Step 3: Formulating a Gödel sentence

Goal of this section: To show how a Gödel sentence for a formal system can be constructed, given the result of the previous section.

The direct approach: What about something like

~ϕ(3097540239750934309)

where 3097540239750934309 is the Gödel # of "~ϕ(3097540239750934309)"?

The substitution operation:

Removing all occurrences of a given free variable in a formula, and replacing them with the symbol for a specific number.

Examples:

Formula	Variable letter to be replaced	Number symbol to replace it with	Result
x = ssy	x	s0	s0 = ssy
(∃x) x = ssy	y	sss0	(∃x) x = sssss0
y + sy = ss0	y	s0	s0 + ss0 = ss0
⋮	⋮	⋮	⋮

The Sub function:

The mathematical function that takes the Gödel # of a formula, the Gödel # of a variable letter, and a third number as inputs, and gives as output: the Gödel # of the formula that results from substituting the symbol for the third number for all occurrences of the variable in the formula.

Formula	Variable	Number	Result
x = ssy	x	s0	s0 = ssy
(∃x) x = ssy	y	sss0	(∃x) x = sssss0
y + sy = ss0	y	s0	s0 + ss0 = ss0
⋮	⋮	⋮	⋮

Important:

Sub(65,4,8) is a number.

“**Sub**(65,4,8)” is an expression in the formal system (where “**Sub**” is the formal system’s representation of the Sub function).

“**Sub**(65,4,8)” refers to the number, Sub(65,4,8).

So, we have:

Inputs of Sub function			Outputs of Sub function
$2^{11} \cdot 3^{8} \cdot 5^{7} \cdot 7^{7} \cdot 11^{13}$	11	1	$2^{7} \cdot 3^{6} \cdot 5^{8} \cdot 7^{7} \cdot 11^{7} \cdot 13^{13}$
$2^{1} \cdot 3^{3} \cdot 5^{11} \cdot 7^{2} \cdot 11^{11} \cdot 13^{8} \cdot 17^{7} \cdot 19^{7} \cdot 23^{13}$	13	3	$2^{1} \cdot 3^{3} \cdot 5^{11} \cdot 7^{2} \cdot 11^{11} \cdot 13^{8} \cdot 17^{7} \cdot 19^{7} \cdot 23^{7} \cdot 29^{7} \cdot 31^{7} \cdot 37^{6}$
$2^{13} \cdot 3^{9} \cdot 5^{7} \cdot 7^{13} \cdot 11^{8} \cdot 13^{7} \cdot 17^{7} \cdot 19^{13}$	13	1	$2^{7} \cdot 3^{6} \cdot 5^{9} \cdot 7^{7} \cdot 11^{7} \cdot 13^{6} \cdot 17^{8} \cdot 19^{7} \cdot 23^{7} \cdot 29^{6}$
⋮	⋮	⋮	⋮

How to find the value of Sub(x,y,z):

a) Find the wff with Gödel number x.
b) Find the variable with Gödel number y.
c) Find the symbol that represents the number z in the formal system.
d) In the wff mentioned in (a): take all occurrences of the variable mentioned in (b), and substitute the symbol mentioned in (c).
e) Then find the Gödel # of the resulting sentence.

Some interesting formulas:

~ϕ[**Sub**(y,13,y)] **(1)**

Suppose the Gödel number of formula (1) is *n*. Now consider:

~ϕ[**Sub**(‘*n*’,13,‘*n*’)] **(3)**

What is the value of **Sub**(‘*n*’,13,‘*n*’)?

a) Find the wff with Gödel number *n*. That is formula (1).
b) Find the variable with Gödel number 13. That is “*y*”.
c) Find the symbol that represents the number *n* in the formal system. That is ‘*n*’.
d) In formula (1): Take all occurrences of “y”, and substitute ‘n’. The result is formula (3) itself.
e) So the value of **Sub**(‘*n*’,13,‘*n*’) is the Gödel # of formula (3).

This is interesting:

- Formula (3) then says that *its own* Gödel # does not have property ϕ.
- ϕ is the property that the Gödel #'s of all the theorems of the formal system have.
- So formula (3) says that formula (3) itself is not a theorem of the system.

Conclusion of the proof

Suppose formula (3) is true:

- Then it is true but not a theorem. → The system is incomplete.

Suppose formula (3) is false:

- Then it is false and is a theorem. → The system is unsound.

Gödel's Second Theorem

- No consistent formal system, capable of representing arithmetic, can be used to prove its own consistency.

What this does not say:

- Anything about 'knowledge'
- Anything about proof in the standard English sense.
- Anything about limitations of the human mind
- Anything about the imperfections of mathematics
- That a given system's consistency cannot be proven in *any* formal system. (It can be proven in a stronger system.)