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Chapter 9: Naive Set Theory

To Discuss Today:
- What are sets
- Axioms of naive set theory
- Set theoretic terminology
- Theorems
- What sets aren't.

About Set Theory

A little history
Why it's interesting
- Basis of mathematics?
- Used in defining:
  - Numbers
  - Geometrical objects
  - Functions
  - Probabilities
  - Philosophical objects: properties, propositions
- Used in understanding infinity
- Fun & famous paradoxes

What is a set?

A set is a collection/group?
- *Problem:* Empty set? Singleton sets?

Sets are 'primitive'?
- *Problem:* How are we supposed to have this concept?

Sets are implicitly defined by the axioms of set theory?
- Existence condition
- Identity condition

The Axioms of Naive Set Theory

The Naive Comprehension Axiom

$$(\exists s)(x)\ (x \in s \leftrightarrow \phi(x))$$

- Examples:
  - There is a set of all cats: $(\exists s)(x)\ (x \in s \leftrightarrow Cx)$
  - There is a set of all fat cats: $(\exists s)(x)\ (x \in s \leftrightarrow (Cx \ \&\ Fx))$
  - There is a set containing me and the Empire State Building: $(\exists s)(x)\ (x \in s \leftrightarrow (x{=}m \vee x{=}e))$

The Axiom of Extensionality

$$(s)(r)\ [s{=}r \leftrightarrow (x)\ (x \in s \leftrightarrow x \in r)]$$

- Examples:
  - {2,3}
  - {3,2}
  - the set of all prime numbers less than 5
  - the set of all integers between 1 and 4

Set Theory Terminology

Representing sets:

- $\{a, b, c\}$
- $\{2, 4, 6, ...\}$
- $\{x: Fx\}$ or $\{x| Fx\}$

The empty set:

- $\{\}$, $\varnothing$

The universal set:

- U, V

Singleton set:

- A set with exactly one member.
- Example: $\{2\}$, {Mike}

Union of two sets:

- $s \cup r = \{x: x \in s \vee x \in r\}$
- Example: $\{a, b, c\} \cup \{c, d\} = \{a, b, c, d\}$

Intersection of two sets:

- $s \cap r = \{x: x \in s \;\&\; x \in r\}$
- Example: $\{a, b, c\} \cap \{c, d\} = \{c\}$

Complement of a set:

- $s' = \{x: x \notin s\}$

s minus r:

- $s - r = \{x: x \in s \;\&\; x \notin r\}$
- Example: $\{a, b, c\} - \{c, d\} = \{a, b\}$

Subset:

- $s \subseteq r \leftrightarrow (x)(x \in s \rightarrow x \in r)$
- Example:
  - $\{a, b\} \subseteq \{a, b, c\}$
  - $\{a, b\} \subseteq \{a, b\}$
  - $\{\} \subseteq \{a, b\}$

Proper subset:

- $s \subset r \leftrightarrow [(x)(x \in s \rightarrow x \in r) \;\&\; s \neq r]$
- (Same as subset, except a set is not a proper subset of itself.)

Powerset:

- $\wp s = \{x: x \subseteq s\}$
- Example: $s = \{a, b\}$
- $\wp s = \{\{a\}, \{b\}, \{a,b\}, \{\}\}$

Union of a set of sets:

- $\bigcup s = \{x: (\exists y)\ (y \in s \;\&\; x \in y)\}$
- Example:
  - $s = \{\{a\}, \{a,b\}, \{c\}\}$
  - $\bigcup s = \{a, b, c\}$

Intersection of a set of sets:

- $\bigcap s = \{x: (y)\ (y \in s \rightarrow x \in y)\}$
- Examples:
  - $s = \{\{a\}, \{a,b\}, \{c\}\}$
  - $\bigcap s = \{\}$

$r = \{\{b\}, \{a,b\}, \{c,b\}\}$

$\bigcap r = \{b\}$

Disjoint sets:

r and s are disjoint when $r \cap s = \varnothing$.

Example: {a, b} and {c} are disjoint.

Open, closed, and half-open intervals:

$(a, b) = \{x: a < x < b\}$

$[a, b] = \{x: a \leq x \leq b\}$

$[a, b) = \{x: a \leq x < b\}$

$(a, b] = \{x: a < x \leq b\}$

Example: [0,1) is the set containing all real numbers from 0 up to 1 (including 0 but not including 1).

Terms vs. formulas:

Terms: $s \cup r$, $s \cap r$, s', s - r, $\wp s$, $\bigcup s$, $\bigcap s$

Formulas: $s \subseteq r$, $s \subset r$

Theorems

Theorem 1:*

Given any open sentence, ϕ (with one free variable), there is exactly one set whose members are all and only the objects satisfying ϕ.

*This is not really true.

Theorem 2:

There is an empty set, i.e., a set with no members.

Theorem 3:*

There is a universal set, i.e., a set of which everything is a member.

Theorem 4:

Every set is a subset of itself: (s) $s \subseteq s$.

Theorem 5:

For any sets, s and r, s=r iff ($s \subseteq r$ and $r \subseteq s$).

Theorem 6:

Every pair of sets has a unique union, i.e., for all s, r, $s \cup r$ exists and is unique.

Theorem 7:

Every pair of sets has a unique intersection.

Theorem 8:*

Every set has a unique complement.

Theorem 9:

Every set has a unique powerset.

Theorem 10:

Every object has a singleton set, i.e., for all x, there exists the set {x}.

Theorem 11:

For every x, y, there exists the set {x, y}.

What Sets Are Not

Not aggregates/mereological sums

Not properties

Sets are defined extensionally

Properties are not

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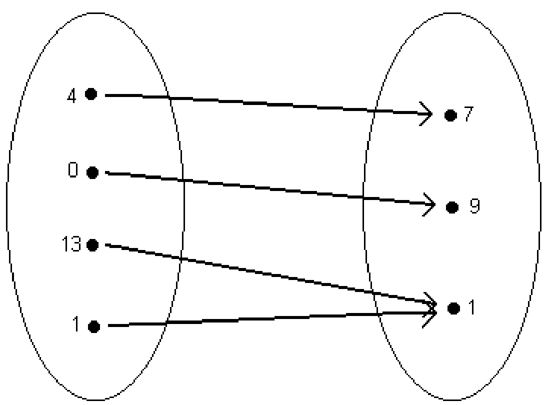
Chapter 10: Applications of Set Theory

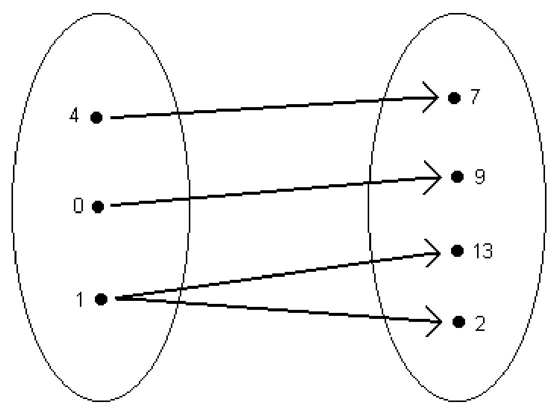
To Discuss Today:
Ordered pairs
Functions
Natural numbers
Infinity

Ordered Pairs
Like sets, but order matters
$\langle a, b\rangle = \{\{a\}, \{a, b\}\}$
$\langle a, b, c\rangle = \langle a, \langle b, c\rangle\rangle$
$\langle x_1, x_2, \ldots x_n\rangle = \langle x_1, \langle x_2, \ldots x_n\rangle\rangle$

Functions
Exactly one output for each input
Example: $y = x^2 + 4$:
y is a function of *x*
x not a function of *y*



A function. Notice how each item on the left (in the domain) has one arrow pointing away from it.



Not a function. Notice how one of the items on the left (in the domain) has two arrows pointing away from it.

Terminology:
argument(s)
values
domain
range
"from", "onto", "into"
Functions can also have multiple inputs.

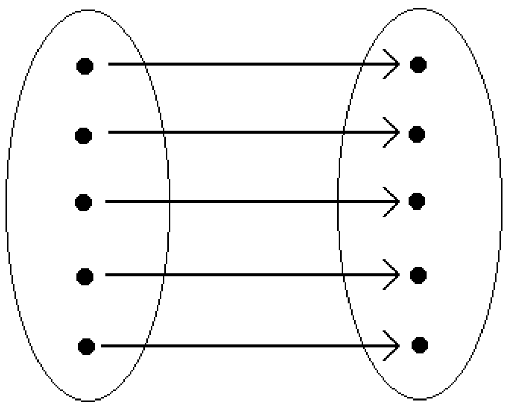
Example:
List the functions from {a, b} onto {c, d}
a → c
b → d

a → d
b → c

Not these (not *onto* {c,d}):
a → c
b → c

a → d
b → d

One-one function:
Every input is paired with a unique output, and vice versa



A one-one function. Each item on the left is correlated with one item on the right, and vice versa.

Natural Numbers

Numbers can be 'reduced' to set theory

Russell's approach:

Intuitive idea:
0 = the set of all 0-membered (empty) sets
1 = the set of all single-membered sets
2 = the set of all 2-membered sets
etc.

How to say that without using any number words?

x has the same cardinality as y:
There is a one-one function from x onto y. Also called: x and y are equinumerous

x has a lower cardinality than y:
There is no one-one function from x onto y, but there is a one-one function from x onto a subset of y.

x has a higher cardinality than y:
There is no one-one function from x onto y, but there is a one-one function from a subset of x onto y.

The numbers, again:
0 = the set of all sets that are equinumerous with {}
1 = the set of all sets that are equinumerous with {0}

2 = the set of all sets that are equinumerous with {0,1}
3 = the set of all sets that are equinumerous with {0,1,2}
etc.

Other ways of doing it?
Frege: uses ‘concepts’ instead of sets
Alternate way:
0 = {}
1 = {{}}
2 = {{{}}}
etc.

Philosophical question: Are these plausible accounts of numbers?

Countable Infinities

ω, $\aleph_0$:
The set of all sets that are equinumerous with {0, 1, 2, 3, ...}.
= The cardinality of the set of natural numbers
This is the first infinite “number”.
Also: it is a ‘countable infinity’.

More countably infinite sets:
{1, 2, 3, ...}
(consider *f*(x) = x + 1.)
{0, 2, 4, ...}
(consider *f*(x) = 2x.)
{... -2, -1, 0, 1, 2, ...}

0	1	-1	2	-2	3	-3	...
↕	↕	↕	↕	↕	↕	↕	
0	1	2	3	4	5	6	...

{x: x is prime}, {1, 2, 3, 5, 7, 11, ...}

1	2	3	5	7	11	13	...
↕	↕	↕	↕	↕	↕	↕	
0	1	2	3	4	5	6	...

Interesting characteristic of infinite sets:
An infinite set can be mapped one-one onto a proper subset of itself.

The Continuum

c:
The cardinality of the set of real numbers

c > ω.
The natural #s can be mapped one-one onto a subset of the real #s. (obvious)
They cannot be mapped one-one onto all of the real #s. Cantor’s “Diagonalization Argument”:
Assume *f* is a one-one function from the natural #s onto the real #s between 0 and 1.

x	*f(x)*
0	. <u>5</u> 4 5 0 9 2 ...
1	. 4 <u>3</u> 6 2 1 4 ...
2	. 1 9 <u>7</u> 9 6 7 ...

3	. 8 4 9 4 6 5 ...
4	. 4 6 5 5 9 6 ...
5	. 6 5 4 6 5 0 ...
⋮	⋮

We can construct a real #, R, that is not one of the values of *f*.

x	*f(x)*	Digits of R
0	. 5 4 5 0 9 2 ...	. 6
1	. 4 3 6 2 1 4 ...	. 6 4
2	. 1 9 7 9 6 7 ...	. 6 4 8
3	. 8 4 9 4 6 5 ...	. 6 4 8 5
4	. 4 6 5 5 9 6 ...	. 6 4 8 5 0
5	. 6 5 4 6 5 0 ...	. 6 4 8 5 0 1
⋮	⋮	⋮

Therefore, *f* is not a one-one function from the natural #s onto the real #s (by RAA).
So there is no one-one function from the natural #s onto the real #s (by UG).

Other interesting result: there are many more infinite cardinals.
The 'powerset theorem': the powerset of A always has a higher cardinality than A.
Hence, there is an infinite hierarchy of infinite cardinals.

Philosophical Questions

Aristotle's doctrine
- The impossibility of an 'actual' infinity.

Galileo's argument
- Which is greater: the number of natural numbers, or the number of perfect squares?
- First answer: There are more natural numbers than perfect squares. (Argument: natural numbers include the perfect squares, plus a lot more.)
- Second answer: There are just as many perfect squares as natural numbers. (Argument: for every natural # n, there is a square, n^2.)
- Conclusion: Infinite sets are neither greater, nor less, nor equal to, other infinite sets.
- Further conclusion: Infinity is not a genuine number?

Calculus
- Does not vindicate treatment of infinity as a number
- Standard approach uses only real #s.
  - No infinities
  - No infinitesimals

Cantor's doctrine
- Embraces the "one-one function" test
- Dismisses Galileo's 'first answer' (the natural numbers include the perfect squares, plus a lot more)
- There are infinite numbers, in the same sense that the natural #s are numbers
- Is Cantor right?

Cantor’s conception is a *generalization* and *extension* of the intuitive notion of “greater than”.
Plausibility depends on the reduction of numbers to sets.

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Chapter 11: Less Naive Set Theory

To Discuss:

- Russell's paradox
- Responses:
  - The theory of types
  - New Foundations
  - Von Neumann
  - Zermelo-Fraenkel
- The Axiom of Choice

Russell's Paradox

Let $r = \{x: x \notin x\}$

Question: $r \in r$?

Formally:

1.	$(\exists s)(x)\ (x \in s \leftrightarrow x \notin x)$	Comprehension Axiom
2.	$(x)\ (x \in \underline{r} \leftrightarrow x \notin x)$	1 EI
3.	$\underline{r} \in \underline{r} \leftrightarrow \underline{r} \notin \underline{r}$	2 UI

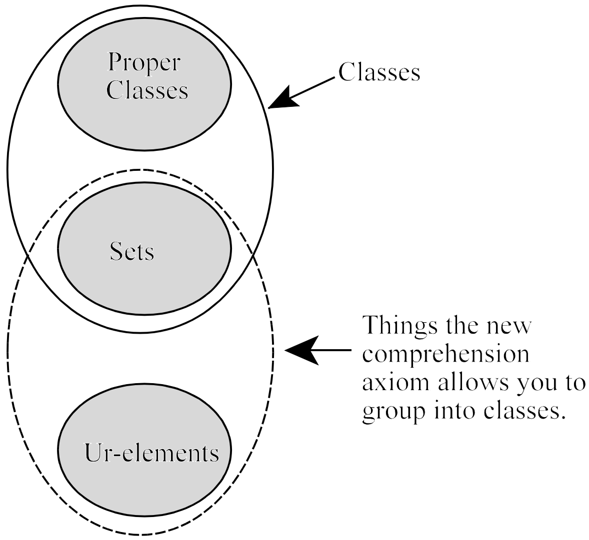
The Theory of Logical Types (Russell)

- Objects organized into a hierarchy
  - Type 0: ur-elements
  - Type 1: sets containing type 0 objects
  - Type 2: sets containing type 1 objects
  - etc.
- Predicates have type restrictions
  - "$\in$": object on right must have higher type than object on left
- Implications:
  - No Russell set
  - No universal set
  - No absolute complement of a set
- A better variant of type theory: cumulative types

New Foundations (Quine)

- Axiom of Comprehension:
  - $(\exists s)(x)\ (x \in s \leftrightarrow \phi(x))$
- holds when ϕ is a stratified predicate. Examples: which of these are stratified?
  - $x \notin x$
  - $x \in x$
  - $x = x$
  - $(y)\ x \in y$
- Implications:
  - No Russell set
  - Allows universal set
  - Allows absolute complement

Von Neumann Set Theory



Divides objects into:
- Classes: sets, proper classes
- Ur-elements

Axiom of comprehension is restricted:
- Only ur-elements and sets can be grouped into classes
  - $(\exists s)(x)\ [x \in s \leftrightarrow (\sim Px \ \&\ \phi(x))]$

Implications:
- No Russell set. Why:

1.	$(\exists s)(x)\ (x \in s \leftrightarrow [\sim Px \ \&\ x \notin x])$	Von Neumann Comprehension Axiom
2.	$(x)\ (x \in \underline{r} \leftrightarrow [\sim Px \ \&\ x \notin x])$	1 EI
3.	$\underline{r} \in \underline{r} \leftrightarrow (\sim P\underline{r} \ \&\ \underline{r} \notin \underline{r})$	2 UI

So $\underline{r}$ is not a member of itself and is a proper class.

Zermelo-Fraenkel Set Theory (ZF or ZFC)

Is a 'pure' set theory. You get:

{}
{{}}
{{}, {{}}}
{{}, {{}}, {{}, {{}}}}
etc.

Axioms:

Axiom of Extensionality:

$(x)(y)\ [(z)(z \in x \leftrightarrow z \in y) \rightarrow x=y]$

Axiom of Separation:

$(x)(\exists y)(z)\ [z \in y \leftrightarrow (z \in x \ \&\ \phi(z))]$

Unordered Pair Axiom:

$(x)(y)(\exists z)(w)\ [w \in z \leftrightarrow (w=x \vee w=y)]$

Union Axiom:

$(x)(\exists y)(z)\ [z \in y \leftrightarrow (\exists w)\ (w \in x \ \&\ z \in w)]$

Powerset Axiom:

$(x)(\exists y)(z)\ [z \in y \leftrightarrow (w)\ (w \in z \rightarrow w \in x)]$

Axiom of Infinity:

$(\exists x)\ [(\exists y)\ (y \in x \ \&\ (z)\ z \notin y)\ \&\ (y)\ (y \in x \rightarrow (\exists z)[z \in x \ \&\ y \in z \ \&\ (w)(w \in z \rightarrow w{=}y)])]$

Axiom of Replacement: For any function, there exists a set containing all its values.

$(x)\ [(y)\ (y \in x \rightarrow (\exists! z)\ \phi(y,z)) \rightarrow (\exists w)(z)\ (z \in w \leftrightarrow (\exists y)\ [y \in x \ \&\ \phi(y,z)])]$

Axiom of Foundation: No set has a nonempty intersection with each of its own elements.

$(x)\ [(\exists y)\ y \in x \rightarrow (\exists y)\ (y \in x \ \&\ \sim(\exists z)\ [z \in y \ \&\ z \in x])]$

Rules out the likes of:

{{{...}}}

A = {B} and B = {A}

The Axiom of Choice

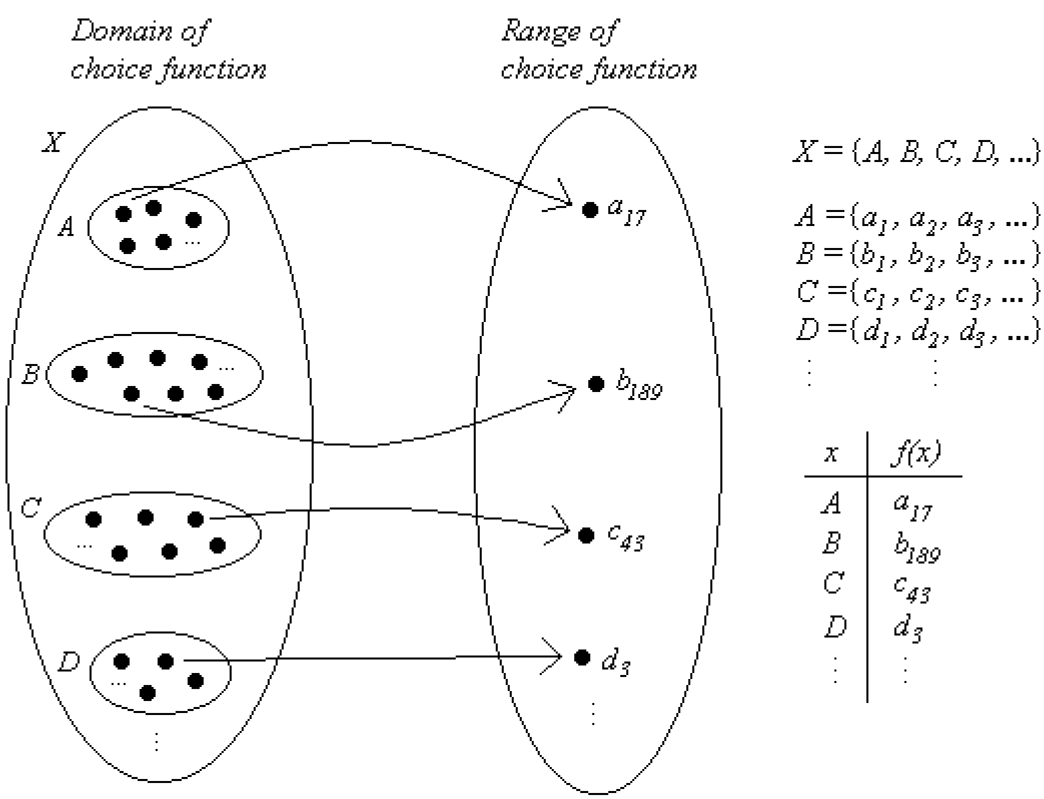
Two formulations:

If *X* is a (non-empty) set of (non-empty) sets, then there exists a function that maps each member of *X* onto a member of itself.

If X is a (non-empty) set of (non-empty, disjoint) sets, there exists a set which contains exactly one member from each member of X.

Intuitive idea: Enables us to ‘choose’ an element from each of the sets in X.

Illustration:



AC is controversial:

Most think it is intuitively obvious.
Some smart people think it is false. (e.g., Borel, Lebesgue, Brouwer)
Philosophical issue: Does a function require a specifiable rule? Does a set require a defining property?

Consequences of AC:
Well-ordering principle
Banach-Tarski paradox

The Independence of AC:
Cannot be proven/disproven in ZF.

The Continuum Hypothesis

The next cardinality above ω is *c*.
This is independent of ZFC.

Philosophical Questions about Sets

Do non-constructible mathematical objects exist?
Do sets exist? Does the empty set exist?
Which version of set theory, if any, is correct?
How to decide whether to accept AC, or the continuum hypothesis?
Is the Frege/Russell reduction of numbers to sets good? Are numbers sets?
What is the best solution to Russell’s Paradox?