Phil 2440
Chapter 5: Predicate Logic Symbolizations

To discuss today:

- Atomic sentences in predicate logic
- Quantifiers
- Some important kinds of sentences
- Well-formed formulas

About the predicate calculus

- What is the predicate calculus?
- Why is the predicate calculus better than the propositional calculus?

Atomic propositions and their structure

- Predicates and subjects
  - "All cats are furry"
  - Logical vs. grammatical predicates & subjects
  - "It is raining"
- Relations
- Symbolizing things
  - Individuals: a, b, ..., t
  - Predicates: A, B, ...
  - Atomic sentences: Fa, Cb, Rca, ...
- Variables & open sentences
  - Individual variables: x, y, z, u, v, w
  - Open sentences: Fx, Rxy, ...

Quantified sentences

- Quantifiers in English. Examples:
  - All cats are furry.
  - Some cats are furry.
  - Most cats are furry.
  - No cats are furry.
- Quantifiers in predicate logic:
  - Universal quantifier: $(\forall x)$, (x), $(\forall y)$, (y), ...
  - Existential quantifier: $(\exists x)$, $(\exists y)$, ...
- Examples:
  - $(\exists x)$ Fx
  - (x) Cx
- Domain of a quantifier:
  - "Drinks for everyone!" Interpretations:
    - (x) Dix
    - (x) ((Px & Rx) $\rightarrow$ Dix)
- Multiple quantifiers:
  - "Someone loves everyone"

= (∃x)(x loves everyone)
= (∃x)(y)(x loves y)
= (∃x)(y) Lxy

Quantifier scope:

A quantifier goes with the first complete sentence following it.

“Bound” vs. “free” variables

Examples:

(∃x) (Cx & Fx)
(∃x) Cx & Fx
(∃x) Cx & (x) Fx
(x) Lxy

Important kinds of sentences and how to symbolize them

“All A’s are B” = (x)(Ax → Bx)
“Some A’s are B” = (∃x)(Ax & Bx)
“Some A’s are non-B” = (∃x)(Ax & ~Bx)
“No A’s are B” = (x)(Ax → ~Bx) = ~(∃x)(Ax & Bx)
Existential import
“Only” and “unless”:

“Only A’s are B” = (x)(Bx → Ax) = (x)(~Ax → ~Bx)
“A thing is A unless it is B” = (x)(~Bx → Ax)

Times and places:

“Someday I’ll be famous” = (∃x)(Dx & Fix)
“God is everywhere” = (x)(Px → Lgx)

Well-formed formulas

Include open sentences & complete sentences.

Rules for wff’s:

1. Atomic formulas are wff’s.
2. If “ϕ” is a wff, then “~ϕ” is a wff.
3. If “ϕ” and “ψ” are wff’s then “(ϕ ∨ ψ)”, “(ϕ & ψ)”, “(ϕ → ψ)”, and “(ϕ ↔ ψ)” are wff’s.
4. If “ϕ” is a wff, then “(x) ϕ”, “(∃x) ϕ”, “(y) ϕ”, “(∃y) ϕ”, etc., are wff’s.

Examples:

(Hx → P)	~Acac
(x) y → Fx	(Ha ∨ Fy)
Ax & Fy ∨ Ba	(Ha ∨ (x)Fy)
Ca (∃x)	(∃y)(z) Ax

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Chapter 6: Predicate Logic Proofs

To discuss today:

5 new inference rules
Strategy for predicate logic proofs

Applying the old rules to predicate logic sentences

All the old rules still apply.
Implicational rules: only apply to whole lines.
Examples: which of the following are good?

Example 1:

1. (∃x) Fx → Ga
2. (∃x) Fx
3. Ga   1,2 MP

Example 2:

1. (∃x) (Fx → Ga)
2. (∃x) Fx
3. Ga   1,2 MP

Example 3:

1. (∃x) (Fx ∨ Gx)
2. (∃x) ~Fx
3. (∃x) Gx   1,2 DS

Example 4:

1. (x) (~Fx & ~Gx)
2. (x) ~(Fx ∨ Gx)   1 DeM

Quantifier Negation (QN)

Rule: One kind of quantifier can be switched to the other kind, while adding/subtracting a "~" on both sides of it. Thus:

(x) ϕ ≡ ~(∃x) ~ϕ
(∃x) ϕ ≡ ~(x) ~ϕ
(∃x) ~ϕ ≡ ~(x) ϕ
(x) ~ϕ ≡ ~(∃x) ϕ

Example:

~(∃x) Fx → ~(∃x) ~Gx
(x) ~Fx
∴ (x) Gx

1. ~(∃x) Fx → ~(∃x) ~Gx   p
2. (x) ~Fx   p
3. (x) ~Fx → ~(∃x) ~Gx   1 QN
4. ~(∃x) ~Gx   2,3 MP
5. (x) Gx   4 QN

Existential Instantiation (EI)

Rule: Remove existential quantifier and substitute for every variable under its scope an unknown symbol:

$$\frac{(\exists v)\ \phi(v)}{\phi(\underline{u})}$$

Using ‘unknown’ symbols.

Examples:

Example 1:

1. $(\exists x)\ (Fx \ \&\ Gx)$
2. $F\underline{a} \ \&\ G\underline{a}$ 1 EI

Example 2:

1. $(\exists x)\ (Fx \ \&\ Gx)$
2. $F\underline{b} \ \&\ G\underline{b}$ 1 EI

Example 3:

1. $(\exists x)\ [Fx \ \&\ (y)\ (Fy \rightarrow Gx)]$
2. $F\underline{a} \ \&\ (y)\ (Fy \rightarrow G\underline{a})$ 1 EI

Restriction on EI: $\underline{u}$ cannot appear previously in the proof

Example 4:

1. $(\exists x)\ Hx$ p
2. $(\exists x)\ Px$ p
3. $H\underline{a}$ 1 EI
4. $P\underline{a}$ 2 EI
5. $H\underline{a} \ \&\ P\underline{a}$ 3,4 conj.

Example 5:

1. $(\exists x)\ Hx$ premise
2. $(\exists x)\ Px$ premise
3. $H\underline{a}$ 1 EI
4. $P\underline{b}$ 2 EI

Note: cannot apply EI to part of a line.

Example 6:

1. $(\exists x)\ (Fx \ \&\ Gx) \vee (\exists y)\ Ay$
2. $(F\underline{a} \ \&\ G\underline{a}) \vee (\exists y)\ Ay$ 1 EI

Example 7:

1. $(x)(\exists y)\ Myx$
2. $(x)\ M\underline{b}x$ 1 EI

Existential Generalization (EG)

Rule: Replace one or more occurrences of a constant/unknown with a variable, and add an existential quantifier to the sentence.

$$\frac{\phi(\underline{u})}{(\exists v)\ \phi(v)} \qquad \frac{\phi(c)}{(\exists v)\ \phi(v)}$$

Example 1:

1. Fc
2. $(\exists x)\ Fx$ 1 EG

Example 2:

1. Fa & Ga
2. (∃x) (Fx & Gx)   1 EG

Example 3:

1. Fa & Ga
2. (∃x) (Fx & Ga)   1 EG

Universal Instantiation (UI)

Rule: Remove a universal quantifier and replace all variables under its scope with a constant/unknown symbol.

$$\frac{(v)\ \phi(v)}{\phi(\underline{u})} \qquad \frac{(v)\ \phi(v)}{\phi(c)}$$

Example:

1. (x) Fx
2. Fa   1 UI

Universal Generalization (UG)

Rule: Replace every occurrence of an unknown symbol with a variable, and add a universal quantifier to the sentence.

$$\frac{\phi(\underline{u})}{(v)\ \phi(v)}$$

Restrictions:

Does not work on constants.
$\underline{u}$ does not occur previously in a line obtained by EI
$\underline{u}$ does not occur in an undischarged assumption

Examples: which of these are correct uses of UG?

Example 1:

1. (x) (Sx → Dx)   p
2. (x) Sx   p
3. ~D$\underline{a}$ a
4. S$\underline{a}$ → D$\underline{a}$ 1 UI
5. S$\underline{a}$ 2 UI
6. D$\underline{a}$ 4,5 MP
7. D$\underline{a}$ & ~D$\underline{a}$ 3,6 conj.
8. D$\underline{a}$ 3-7 RAA
9. (x) Dx   8 UG

Example 2:

1. (x) (Sx → Dx)   p
2. (x) Sx   p
3. S$\underline{a}$ → D$\underline{a}$ 1 UI
4. S$\underline{a}$ 2 UI
5. D$\underline{a}$ 3,4 MP
6. (x) Dx   5 UG

Example 3:

1. (∃x) Fx — p
2. Fb — 1 EI
3. (x) Fx — 2 UG

Example 4:

1. ~(x) Fx — p
2. →2. Fb — a
3. (x) Fx — 2 UG
4. (x) Fx & ~(x) Fx — 1,3 conj.
5. ~Fb — 2-4 RAA
6. (x) ~Fx — 5 UG

Miscellaneous Stuff

Remembering the names of the rules.
General predicate-logic proof strategy.
Use EI first (before UI).
Example 1:

(x) (Ax ∨ Bx)
(x) (Bx → Ax)
∴ (x) Ax

1.	(x) (Ax ∨ Bx)	p
2.	(x) (Bx → Ax)	p
3.	Aa ∨ Ba	1 UI
4.	Ba → Aa	2 UI
5.	~Aa	a
6.	~Ba	4,5 MT
7.	Aa	3,6 DS
8.	Aa & ~Aa	7,5 conj
9.	Aa	5-8 RAA
10.	(x) Ax	9 UG

Example 2:

(∃x)Ax → (x)(Bx → Cx)
Am & Bm
∴ Cm

1.	(∃x)Ax → (x)(Bx → Cx)	p
2.	Am & Bm	p
3.	Am	2 simp
4.	Bm	2 simp
5.	(∃x) Ax	3 EG
6.	(x) (Bx → Cx)	1,5 MP
7.	Bm → Cm	6 UI
8.	Cm	4,7 MP

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Chapter 7: Relations and Identity

To Discuss Today:
Logical properties of relations
Symbolizations involving identity
Logical laws of identity

Properties of relations
Symmetry:
Symmetric: Rxy ⊢ Ryx.
"x is next to y"
Asymmetric: Rxy ⊢ ~Ryx
"x is bigger than y"
Non-symmetric: (neither symmetric nor asymmetric)
"x hits y"
Transitivity:
Transitive: Rxy & Ryz ⊢ Rxz
"x is bigger than y"
Intransitive: Rxy & Ryz ⊢ ~Rxz
"x is the daughter of y"
Non-transitive: (neither transitive nor intransitive)
"x is a friend of y"
Reflexivity:
Reflexive: Rxy ⊢ Rxx
"x lives in the same house as y"
Irreflexive: ~Rxx
"x is older than y"
Non-reflexive: (neither reflexive nor irreflexive)
"x loves y"
Equivalence relations:
Have the properties of 'equivalence':
Reflexive, symmetric, transitive

Identity
General points about identity:
Numerical vs. type identity
Properties of identity:
Equivalence relation
Symbol for identity:
"x = y": x is numerically identical with y
Note: "=" is a 2-place predicate.
Numerical statements:
There is exactly 1 cat:
(∃x)[Cx & (y)(Cy → y=x)]

There are exactly 2 cats:

$(\exists x)(\exists y)\ [(Cx \ \& \ Cy) \ \& \ x \neq y \ \& \ (z)\ (Cz \rightarrow [z=x \lor z=y])]$

There are exactly 3 cats:

$(\exists x)(\exists y)(\exists z)\ [(Cx \ \& \ Cy \ \& \ Cz) \ \& \ (x \neq y \ \& \ y \neq z \ \& \ x \neq z) \ \& \ (w)(Cw \rightarrow (w=x \lor w=y \lor w=z))]$

There is at most 1 cat:

$\sim(\exists x)(\exists y)\ [(Cx \ \& \ Cy) \ \& \ x \neq y]$

Definite descriptions:

The King of France is bald:

$(\exists x)[Kxf \ \& \ (y)(Kyf \rightarrow y=x)] \ \& \ (x)(Kxf \rightarrow Bx)$

$(\exists x)[(Kxf \ \& \ Bx) \ \& \ (y)(Kyf \rightarrow y=x)]$

Aside: Strawson's criticism of Russell's analysis

Logical laws of identity

The Law of Identity (Id):

Intuitive statement: everything is identical to itself.

Rule: Write down "$\alpha=\alpha$" at any stage, where α is any constant or unknown symbol.

Leibniz' Law (LL):

If x=y, then any property of x is a property of y and vice versa.

Rule: From "$\phi(\alpha)$" and "$\alpha=\beta$", deduce "$\phi(\beta)$".

Example:

To prove: $(x)(y)(x=y \rightarrow y=x)$. (Identity is symmetric.)

1. $\underline{a}=\underline{b}$ — a. (for CP)
2. $\underline{a}=\underline{a}$ — Id
3. $\underline{b}=\underline{a}$ — 1,2 LL
4. $\underline{a}=\underline{b} \rightarrow \underline{b}=\underline{a}$ — 1-3 CP
5. $(y)\ (\underline{a}=y \rightarrow y=\underline{a})$ — 4 UG
6. $(x)(y)\ (x=y \rightarrow y=x)$ — 5 UG