CLT for $L^p$ moduli of continuity of Gaussian processes

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Abstract

Let $G = \{G(x), x \in \mathbb{R}^1\}$ be a mean zero Gaussian processes with stationary increments and set $\sigma^2(|x - y|) = E(G(x) - G(y))^2$. Let $f$ be a symmetric function with $Ef(\eta) < \infty$, where $\eta = N(0,1)$. When $\sigma^2(s)$ is concave or when $\sigma^2(s) = s^r, 1 < r \leq 3/2$

$$\lim_{h \downarrow 0} \int_a^b f \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) dx - (b-a)Ef(\eta) \xrightarrow{law} N(0,1)$$

where $\Phi(h, \sigma(h), f, a, b)$ is the variance of the numerator. This result continues to hold when $\sigma^2(s) = s^r, 3/2 < r < 2$, for certain functions $f$, depending on the nature of the coefficients in their Hermite polynomial expansion.

The asymptotic behavior of $\Phi(h, \sigma(h), f, a, b)$ at zero, is described in a very large number of cases.

1 Introduction

Let $G = \{G(x), x \in \mathbb{R}^1\}$ be a mean zero Gaussian process with stationary increments, and set

$$E(G(x) - G(y))^2 = \sigma^2(x-y) = \sigma^2(|x-y|).$$
Clearly $\sigma^2(0) = 0$. To avoid trivialities we assume that $\sigma^2(h) \neq 0$.

When $G$ is continuous and $\sigma^2(h)$ is concave for $h \in [0, h_0]$ for some $h_0 > 0$ and satisfies some other very weak conditions, or when $\sigma^2(h) = h^r$, $1 < r < 2$, for $h \in [0, h_0]$, we show in [4] that

$$\lim_{h \downarrow 0} \int_a^b \left| \frac{G(x + h) - G(x)}{\sigma(h)} \right|^p dx = E|\eta|^p(b - a) \quad (1.2)$$

for all $a, b \in \mathbb{R}$, almost surely, where $\eta$ is a normal random variable with mean zero and variance one, (sometimes also denoted by $N(0, 1)$).

Obviously, the right-hand side of (1.2) is the expected value of the integral on the left-hand side for all $h > 0$. Thus one can think of (1.2) as a Strong Law of Large Numbers for the functional

$$\int_a^b \left| \frac{G(x + h) - G(x)}{\sigma(h)} \right|^p dx. \quad (1.3)$$

It is natural to ask if this functional also satisfies a Central Limit Theorem because this would give the next order in the description of the asymptotic behavior of (1.3).

We consider this question in a more general setting. Fix $-\infty < a < b < \infty$. Let $d\mu(x) = (2\pi)^{-1/2} \exp(-x^2/2) \, dx$ denote standard Gaussian measure on $\mathbb{R}$. For any symmetric function $f \in L^2(\mathbb{R}, d\mu)$, i.e., $Ef(\eta) < \infty$, define

$$I(f, h) = I_G(f, h; a, b) = \int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) \, dx. \quad (1.4)$$

We obtain CLTs for the functionals $I(f, h)$. Clearly they apply to (1.3) by taking $f(\cdot) = |\cdot|^p$.

**Theorem 1.1** Assume that either $\sigma^2(h)$ is concave or that $\sigma^2(h) = h^r$, $1 < r \leq 3/2$. Then for all symmetric functions $f \in L^2(\mathbb{R}, d\mu)$

$$\lim_{h \downarrow 0} \frac{I_G(f, h; a, b) - (b - a)Ef(\eta)}{\sqrt{\text{Var} I_G(f, h; a, b)}} \xrightarrow{\text{law}} N(0, 1). \quad (1.5)$$

When $\sigma^2(h) = h^r$, $3/2 < r < 2$ we no longer get (1.5) for all symmetric $f \in L^2(\mathbb{R}, d\mu)$. However, we do get it for certain $f \in L^2(\mathbb{R}, d\mu)$ depending on the coefficients of the Hermite polynomial expansion of $f$. Let
\( \{H_m(x)\}_{m=0}^{\infty} \) denote the Hermite polynomials. (They are an orthonormal basis for \( L^2(R^1, d\mu) \).) Then for symmetric \( f \in L^2(R^1, d\mu) \),

\[
  f(x) = \sum_{m=0}^{\infty} a_{2m} H_{2m}(x) \quad \text{in} \quad L^2(R^1, d\mu),
\]

where

\[
  a_{2m} = \int f(x) H_{2m}(x) \, d\mu(x)
\]

and

\[
  \sum_{m=0}^{\infty} a_{2m}^2 = \int |f(x)|^2 \, d\mu(x) < \infty.
\]

**Theorem 1.2** Let \( f \in L^2(R^1, d\mu) \) be symmetric and let

\[
  k_0 = \inf_{m \geq 1} \{ m | a_{2m} \neq 0 \}.
\]

Assume that \( \sigma^2(h) = h^r, \) \( 0 < r \leq 2 - 1/(2k_0) \). Then (1.5) holds.

Clearly, Theorem 1.1 contains this result when \( k_0 = 1 \) but not when \( k_0 > 1 \). We can show that when \( f \in L^2(R^1, d\mu) \) is symmetric and its Hermite polynomial expansion is such that (1.9) holds and \( \sigma^2(h) = h^r, \) \( r > 2 - 1/(2k_0) \), left-hand side of (1.5) converges to a \( 2k_0 \)-th order Gaussian chaos. We plan to address this in a subsequent paper.

Theorems 1.1 and 1.2 are consequences of the following general CLT for \( I_G(f, h; a, b) \) and its simple corollary, Corollary 2.1. For \( x, y \in R^1 \) let

\[
  \rho_h(x, y) = \frac{1}{\sigma^2(h)} E(G(x + h) - G(x))(G(y + h) - G(y))
\]

\[= \frac{1}{2\sigma^2(h)} \left( \sigma^2(x - y + h) + \sigma^2(x - y - h) - 2\sigma^2(x - y) \right) \]

\[:= \rho_h(x - y) = \rho_h(y - x). \]

**Theorem 1.3** Assume that for all \( j \in N \)

\[
  \sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^j \, dy \leq C_j \int_a^b \int_a^b |\rho_h(x - y)|^j \, dx \, dy
\]
where $C_j$ is a constant which can depend on $j$. Assume, furthermore, that for all $j \in \mathbb{N}$

$$
\left( \int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \right)^{1/j} = o \left( \int_a^b \int_a^b |\rho_h(x-y)|^{j+1} \, dx \, dy \right)^{1/(j+1)}.
$$

(1.12)

Then for all symmetric functions $f \in L^2(R^1, d\mu)$

$$
\lim_{h \downarrow 0} \frac{I_G(f, h; a, b) - (b - a)Ef(\eta) \sqrt{\text{Var} \, I_G(f, h; a, b)}}{\text{Var} \, I_G(f, h; a, b)} \xrightarrow{\text{law}} N(0, 1).
$$

(1.13)

To complete this analysis we need to describe the behavior of $\text{Var} \, I_G(f, h; a, b)$ as $h$ decreases to zero. We do this in Sections 3 and 4, with varying degrees of precision, depending on the the function $\sigma^2(h)$. We show on page 14 that

$$
\text{Var} \, I_G(f, h; a, b) = \sum_{k=1}^{\infty} a_{2k}^2 \int_a^b \int_a^b (\rho_h(x-y))^{2k} \, dx \, dy.
$$

(1.14)

The following table gives the behavior of the integrals in (1.14) as $h$ decreases to zero for many examples of $\sigma^2(h)$. 
<table>
<thead>
<tr>
<th>$\sigma^2(h)$</th>
<th>$\int_a^b \int_a^b (\rho_h(x - y))^k , dx , dy$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $h^r$, $r &gt; 2 - 1/k$</td>
<td>$\sim C_{1,k} h^{(2-r)k}$</td>
</tr>
<tr>
<td>2) $h$</td>
<td>$\sim \frac{2(b-a)}{2k+1} h$</td>
</tr>
<tr>
<td>3) $h^r$, $r = 2 - 1/k$, $k \ge 2$</td>
<td>$\sim C_{3,k} h \log 1/h$</td>
</tr>
<tr>
<td>4) $h^r$, $0 &lt; r &lt; 2 - 1/k$</td>
<td>$\sim C_{4,k} h$</td>
</tr>
<tr>
<td>5) concave</td>
<td>$\approx h$</td>
</tr>
<tr>
<td>6) regularly varying</td>
<td>$\approx \frac{h}{(\log 1/h)^{k(1-\gamma)}}$</td>
</tr>
<tr>
<td>strictly positive index</td>
<td>$\approx \frac{h}{(\log 1/h)^k}$</td>
</tr>
</tbody>
</table>

where

$$C_{1,k} = \frac{2r^k |r - 1|^k (b-a)^{(r-2)k+2}}{2^k ((r-2)k + 1) ((r-2)k + 2)}$$

$$C_{3,k} = 2(b-a) \frac{|r(r-1)|^k}{2}$$

$$C_{4,k} = 2(b-a) \int_0^\infty \frac{|s+1|^r + |s-1|^r - 2s^r}{2} s^{k-1} \, ds.$$

We use $f \approx g$ at zero, and say that $f$ is approximately equal to $g$ at zero, to indicate that there exists constants $0 < C_1 \le C_2 < \infty$ such that $C_1 \le \liminf_{x \to 0} \frac{f(x)}{g(x)} \le \limsup_{x \to 0} \frac{f(x)}{g(x)} \le C_2$, and $f \sim g$ at zero, and say that $f$ is...
asymptotic to \( g \) at zero, to indicate that there exists a constant \( 0 < C < \infty \) such that \( \lim_{x \to 0} \frac{f(x)}{g(x)} = C \). Analogous definitions apply at infinity.

In order to use Table 1 for a given \( f \in L^2(\mathbb{R}^1, d\mu) \) it is necessary to know \( k_0 \) in (1.9). For the functionals in (1.3), which were the motivation for this paper, \( k_0 = 1 \), since for these functionals \( a_2 = E(|\eta|^p|\eta^2 - 1|)/\sqrt{2} > 0 \). We get the following immediate corollary of Theorem 1.1.

**Corollary 1.1** Let \( G = \{G(x), x \in [a, (1 + \epsilon)b]\} \), for some \( \epsilon > 0 \), be a Gaussian process with stationary increments with increments variance \( \sigma^2(h) \) that is concave on \([0, 2(b - a)]\), or satisfies \( \sigma^2(h) = h^r \), \( 1 < r \leq 3/2 \), on \([0, 2(b - a)]\). Then for all \( p \geq 1 \),

\[
\lim_{h \to 0} \frac{\int_a^b \frac{|G(x+h) - G(x)|^p}{\sigma(h)} \, dx - E|\eta|^p(b - a)}{\sqrt{\Phi(h)}} \xrightarrow{law} N(0, 1), \tag{1.16}
\]

where \( \Phi(h) \) is the variance of the numerator.

The following table gives the asymptotic behavior of \( \Phi(h) \) at zero for different values of \( \sigma^2(h) \):
with the additional condition, in the final expression, that $h \frac{d}{dh} \sigma^2(h)$ is increasing. It is easy to see that the last entry in this table agrees with (6) and (7), with $k = 2$, in Table 1.

In [6, Theorem 2.2] Sodin and Tsirelson give a general CLT for Gaussian functionals which gives some, but not all, of the cases covered by Theorem 1.3. Their theorem states that (1.13) holds whenever

$$
\lim_{h \to 0} \sup_{a \leq x \leq b} \int_a^b |\rho_h(x-y)| \, dy = 0
$$

(1.17)

and for all $k \in N$

$$
\liminf_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x-y)|^{2k} \, dx \, dy}{\sup_{a \leq x \leq b} \int_a^b |\rho_h(x-y)| \, dy} > 0.
$$

(1.18)

When $f$ is increasing on $[0, \infty)$ it suffices to have (1.18) for $k = 1$.  

<table>
<thead>
<tr>
<th>$\sigma^2(h)$</th>
<th>$\Phi(h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $h^{3/2}$</td>
<td>$\sim \left( \frac{b-a}{\sqrt{2}} \left( \frac{3}{8} \right)^2 (E(</td>
</tr>
<tr>
<td>(2) $0 &lt; r &lt; 3/2$</td>
<td>$\sim 2(b-a) h \sum_{k=1}^{\infty} \left( (E(</td>
</tr>
<tr>
<td>(3) $h$</td>
<td>$\sim 2(b-a) h \sum_{k=1}^{\infty} \left( E(</td>
</tr>
<tr>
<td>concave</td>
<td>$\approx h$</td>
</tr>
<tr>
<td>strictly positive index</td>
<td></td>
</tr>
<tr>
<td>concave slowly varying</td>
<td>$\approx \left( \frac{h \sigma'(h)}{\sigma(h)} \right)^2 h$</td>
</tr>
</tbody>
</table>
For all the examples in Table 1 we have that for all $k \in \mathbb{N}$
\[
\sup_{a \leq x \leq b} \int_a^b |\rho_h(x-y)|^k \, dy \approx \int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy \quad (1.19)
\]
so that (1.17) holds for all these examples and condition (1.18) for $k = 1$ is equivalent to
\[
\liminf_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x-y)|^2 \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x-y)| \, dx \, dy} > 0. \quad (1.20)
\]
It is easily seen that this holds in case 5) of Table 1 but not in cases 6) and 7), nor when $\sigma^2(h) = h^r$ for $1 < r \leq 3/2$. Actually, the CLT in [6, Theorem 2.2], as it applies to $I_G(f,h,a,b)$, is contained in Theorem 1.3. (See Remark 5.1.)

It should be clear that we cannot get classical CLTs for $I_G(f,h;a,b)$ for all Gaussian process. For example when $\sigma^2(h) = h^2$
\[
\int_a^b \int_a^b |\rho_h(x-y)|^{2k} \, dx \, dy = 2^k (b-a)^2, \quad (1.21)
\]
so that $\lim_{h \downarrow 0} \text{Var } I_G(f,h;a,b) \neq 0$. To make this example more explicit suppose that Gaussian process $G$ is integrated Brownian motion, then
\[
\lim_{h \downarrow 0} \int_a^b \frac{|G(x+h) - G(x)|^p}{h} \, dx = \int_a^b |B(x)|^p \, dx \quad \text{a. s.} \quad (1.22)
\]
where $B$ is Brownian motion. Obviously, the right-hand side of (1.22) is not $N(0,1)$.

In Section 2 we prove the general Theorem 1.3 and Corollary 2.1. To obtain Theorems 1.1 and 1.2 we must verify that the conditions of Theorem 1.3 and Corollary 2.1 hold, when $\sigma^2(h)$ is concave or when $\sigma^2(h) = h^r$ for $1 < r \leq 2 - 1/(2k_0)$. In Section 3 we do this for $\sigma^2(h)$ concave and in Section 4 when $\sigma^2(h)$ is a power. We give the proofs of Theorems 1.1 and 1.2 in Section 5 and also point out how we obtain the estimates in Tables 1 and 2.

2 Proof of Theorem 1.3

Let $\phi_h(x-y) = \sigma^2(h) \rho_h(x-y)$. Note that
\[
\phi_h(x-y) = \frac{1}{2} \left( \sigma^2(x-y+h) + \sigma^2(x-y-h) - 2\sigma^2(x-y) \right) \quad (2.1)
\]
The $2k$-th Wick product for a mean zero Gaussian random variable $Z$ is
\[ :Z^{2k} := \sum_{j=0}^{k} (-1)^j \binom{2k}{2j} E(Z^{2j}) Z^{2(k-j)}. \tag{2.2} \]

If $Z = N(0,1)$ then $:Z^{2k} := \sqrt{(2^k)!} H_{2k}(Z)$. Hence if $\sigma_Z^2$ denote the variance of $Z$,
\[ :\left( \frac{Z}{\sigma_Z} \right)^{2k} := \sqrt{(2^k)!} H_{2k} \left( \frac{Z}{\sigma_Z} \right). \tag{2.3} \]

**Lemma 2.1** Let $G$ be a mean zero Gaussian process with stationary increments. Assume that
\[ \sup_{a \leq x \leq b} \int_a^b |\rho_h(x-y)|^{2k} dy \leq C \int_a^b |\rho_h(x-y)|^{2k} dx \tag{2.4} \]
and for all $j < 2k$
\[ \sup_{a \leq x \leq b} \left( \int_a^b |\rho_h(x-y)|^j dy \right)^{1/j} = o \left( \int_a^b |\rho_h(x-y)|^{2k} dx dy \right)^{1/2k}. \tag{2.5} \]

Then
\[ \lim_{h \to 0} \int_a^b \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right)^{2k} : dx \overset{law}{=} \sqrt{(2^k)!} N(0,1). \tag{2.6} \]

**Proof** We write
\[ \frac{\int_a^b \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right)^{2k} : dx}{\sqrt{\int_a^b \int_a^b |\rho_h(x,y)|^{2k} dx dy}} = \frac{\int_a^b \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right)^{2k} : dx}{\left( \int_a^b \int_a^b |\rho_h(x,y)|^{2k} dx dy \right)^{1/2}} \tag{2.7} \]
and show that for each $n \geq 1$
\[ \lim_{h \to 0} E \left( \left\{ \int_a^b \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right)^{2k} : dx \right\}^n \right) \]
\[ = \begin{cases} \frac{(2m)!}{2^m m!} \left( \frac{(2k)!}{m!} \right)^m & \text{if } n = 2m \\ 0 & \text{otherwise.} \end{cases} \tag{2.8} \]

Since the right-hand side of (2.8) are the moments of the right-hand side of (2.6) the theorem is proved.
It follows from [5, Lemma 2.2] that

\[ E \left( \prod_{i=1}^{n} (G(x_i + h) - G(x_i))^{2k} : \right) = \sum_{\pi \in \mathcal{P}} \left( \prod_{(i,i') \in \pi} \phi_h(x_i - x_{i'}) \right) \tag{2.9} \]

where the sum runs over all pairings \( \pi \in \mathcal{P} \), the set of pairings of the \( 2kn \) elements which consist of \( 2k \) copies of each of the letters \( x_i, 1 \leq i \leq n \), subject to the restriction that no single letter \( x_i \) is paired with itself.

We say that the letters \( x_i, x_j \) are connected in the pairing \( \pi \) if we can find some sequence \( (i_m, i_{m+1}) \), \( m = 1, \ldots \) of pairs in \( \pi \) with \( i_1 = i, i_p = j \) for some \( p \). By decomposing the set of letters \( x_i, 1 \leq i \leq n \) into connected components we can write (2.9) as

\[ E \left( \prod_{i=1}^{n} (G(x_i + h) - G(x_i))^{2k} : \right) \tag{2.10} \]

\[ = \sum_{l=1}^{\lfloor n/2 \rfloor} \sum_{C_1 \cup C_2 \cup \ldots \cup C_l = \{x_i, i=1,\ldots,n\}} \prod_{j=1}^{l} \sum_{\pi \in \mathcal{P}(C_j)} \left( \prod_{(i,i') \in \pi} \phi_h(x_i - x_{i'}) \right) \]

where the second sum runs over all partitions of \( \{x_i, i=1,\ldots,n\} \) into \( l \) sets, \( C_1, \ldots, C_l \) with \( |C_i| \geq 2, i = 1,\ldots,l \). (\( |C| := \# \) of elements in \( C \).) The third sum runs over all pairings \( \pi \in \mathcal{P}(C_j) \), the set of pairings of the set of \( 2k|C_j| \) elements which consists of \( 2k \) copies of each of the letters \( x_i \in C_j \), subject to the following two restrictions:

(i) no single letter \( x_i \) is paired with itself;

(ii) for any partition \( C_j = A \cup B \), at least one letter of \( A \), is paired with a letter of \( B \). (This condition states that \( C_j \) cannot be further decomposed into connected components.)

We show below that the only non-zero terms of the left-hand side of (2.8) comes when \( n = 2m \) and the partitions have \( m \) parts, \( (C_1, C_2, \ldots, C_m) \), in which case all parts necessarily have two elements; that is, from pairings of \( \{x_i, i=1,\ldots,2m\} \). Referring again to (2.9) we see that for each partition of this sort

\[ \prod_{(i,i') \in \pi} \phi_h(x_i - x_{i'}) = \prod_{j=1}^{n} \phi_h^{2k}(x_{i_j} - x_{i'_j}) \tag{2.11} \]

where \( C_j = (i_j, i'_j) \).
Since there are \(\frac{(2m)!}{2^m m!}\) pairings of \(\{x_i, i = 1, \ldots, 2m\}\) and \((2k!)\) ways to arrange the two sets of \(2k\) elements in each pairing, it follows from (2.10) that

\[
E \left( \int_a^b : (G(x + h) - G(x))^{2k} : dx \right)^{2m} = \frac{(2m)!}{2^m m!} (2k!)^m \left( \int_a^b \int_a^b |\phi_h(x - y)|^{2k} dx \, dy \right)^m
\]

\[
+ \sum_{l=1}^{m-1} \sum_{C_1 \cup C_2 \cup \ldots \cup C_l = \{x_i, i = 1, \ldots, 2m\}} \int_{[a,b]^{2m}} \prod_{j=1}^l \sum_{\pi \in \mathcal{P}(C_j)} \left( \prod_{(i,i') \in \pi} \phi_h(x_i - x_{i'}) \right) \prod_{i=1}^{2m} dx_i.
\]

Since the first term to the right of the equal sign in (2.12) gives (2.8), and

\[
\int_{[a,b]^{2m}} \prod_{j=1}^l \sum_{\pi \in \mathcal{P}(C_j)} \left( \prod_{(i,i') \in \pi} \phi_h(x_i - x_{i'}) \right) \prod_{i=1}^{2m} dx_i = \text{o}\left( \left( \int_a^b \int_a^b |\phi_h(x - y)|^{2k} dx \, dy \right)^{|C_p|/2} \right).
\]

To obtain (2.14) choose any pair of letters \(x_i, x_{i'}\) with \((i, i') \in \pi\). Suppose that \(j\) is the number of times that \((i, i')\) occurs in \(\pi\), then we must have \(1 \leq j < 2k\), since if \(j = 2k\) restriction (ii) would be violated. Each variable \(x_r\) on the left-hand side of (2.14) occurs precisely \(2k\) times. Pick such an
$x_r \neq x_i$ or $x_{i'}$ and use the generalized Hölder’s inequality together with (2.4) to obtain the bound

$$\sup_{a \leq d_j \leq b, \forall j} \int_a^b \prod_{j=1}^{2k} \phi_h(x - d_j) \, dx_r$$

(2.15)

$$\leq \sup_{a \leq d_j \leq b, \forall j} \prod_{j=1}^{2k} \left( \int_a^b |\phi_h(x - d_j)|^{2k} \, dx \right)^{1/2k}$$

$$\leq \prod_{j=1}^{2k} \sup_{a \leq d_j \leq b} \left( \int_a^b |\phi_h(x - d_j)|^{2k} \, dx \right)^{1/2k}$$

$$\leq \sup_{a \leq d \leq b} \left( \int_a^b |\phi_h(x - d)|^{2k} \, dx \right)$$

$$\leq K \int_a^b \int_a^b |\phi_h(x - y)|^{2k} \, dx \, dy.$$  

Here $\{d_j\}_{j=1}^{2k}$ represents the different elements $x_j \in C_p$ that $x_r$ is paired with. Several of the $d_j$ may be the same.

We proceed to successively bound the integrals over each $x_p$ with $p \neq i, i'$. Now, however, there may be less than $2^k$ remaining factors containing $x_p$ since some factors may have been bounded at an earlier stage. If, say there are $q$ factors left when we bound $x_p$, then as in (2.15) we obtain

$$\sup_{a \leq d_j \leq b, \forall j} \int_a^b \prod_{j=1}^{q} \phi_h(x_p - d_j) \, dx_p$$

(2.16)

$$\leq (b - a)^{1-q/2k} \sup_{a \leq d_j \leq b, \forall j} \prod_{j=1}^{q} \left( \int_a^b |\phi_h(x_p - d_j)|^{2k} \, dx_p \right)^{1/2k}$$

$$\leq (b - a)^{1-q/2k} \sup_{a \leq d \leq b} \left( \int_a^b |\phi_h(x_p - d)|^{2k} \, dx \right)^{q/2k}$$

$$\leq (b - a)^{1-q/2k} K \left( \int_a^b \int_a^b |\phi_h(x - y)|^{2k} \, dx \, dy \right)^{q/2k}.$$  

Note that the number of pairs in any $\pi \in \mathcal{P}(C_p)$ is $|C_p|k$. Thus we see that after bounding successively all the integrals involving $x_r$ with $r \neq i, i'$ we have for some $1 \leq j < 2^k$

$$\int_{[a,b]|C_p|} \prod_{(i,i')} \phi_h(x_i - x_{i'}) \prod_{x_i \in C_p} \, dx_i$$

(2.17)
where $K' < \infty$ does not depend on $h$. Since by (2.5)
\[
\int_a^b \int_a^b |\phi_h(x_i - x_{i'})|^j \, dx_i \, dx_{i'} = o\left(\int_a^b \int_a^b |\phi_h(x_i - x_{i'})|^{2k} \, dx_i \, dx_{i'}\right)^{j/2k},
\] (2.18)
we get (2.14).

At this point it should be clear that (2.8) is zero when $n$ is odd since any partition of $\{x_i, i = 1, \ldots, n\}$ into $l$ sets, $C_1, \ldots, C_l$ with $|C_i| \geq 2$, $i = 1, \ldots, l$, contains at least one set with three of more elements.

To proceed we need some more information about the Hermite polynomial expansion of functions in $L^2(R^1, d\mu)$. It is clear that
\[
E(f(X)) = \int f(x) \, d\mu(x) = a_0
\] (2.19)
so that
\[
f(X) - E(f(X)) = \sum_{m=1}^{\infty} a_{2m} H_{2m}(X) \quad \text{in} \ L^2(R^1, d\mu).
\] (2.20)

Let $X$ and $Y$ be $N(0, 1)$ and let $(X, Y)$ be a two dimensional Gaussian random variable. Then
\[
E(H_{2m}(X)H_{2n}(Y)) = (E(XY))^{2m} \delta_{m,n}.
\] (2.21)
This follows by setting $Y = \alpha X + (1 - \alpha^2)^{1/2} Z$, where $\alpha = E(XY)$ and $Z$ is $N(0, 1)$ and is independent of $X$, and using the relationship
\[
\sum_{m=0}^{\infty} \frac{\lambda^m}{\sqrt{m!}} H_m(x) = \exp(\lambda x - \lambda^2/2).
\] (2.22)
Consequently, it follows from (2.21) that
\[
\text{Cov}\ (f(X), f(Y)) = \sum_{m=1}^{\infty} a_{2m}^2 (E(XY))^{2m}.
\] (2.23)

For each $h$ we consider the symmetric positive definite kernel $\rho_h(x, y) = \rho_h(x - y)$. Note that by stationarity and the Cauchy–Schwarz inequality
\[
|\rho_h(x - y)| \leq 1 \quad \forall x, y \in R^1.
\] (2.24)
Therefore, by (2.23)

\[ \text{Var} \left( \int_a^b f \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) dx \right) \]

(2.25)

\[ = \int_a^b \int_a^b \text{Cov} \left( f \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right), f \left( \frac{G(y+h) - G(y)}{\sigma(h)} \right) \right) dx dy \]

\[ = \sum_{m=1}^\infty a_{2m}^2 \int_a^b \int_a^b (\rho_h(x-y))^{2m} dx dy. \]

**Proof of Theorem 1.3** Clearly we need only consider \( f \in L^2(R^1, d\mu) \) of the form

\[ f(x) = \sum_{m=k_0}^\infty a_{2m} H_{2m}(x). \] (2.26)

To begin suppose that there are only a finite number of terms in (2.26) so that for some \( k_1 < \infty \)

\[ f(x) = \sum_{m=k_0}^{k_1} a_{2m} H_{2m}(x). \] (2.27)

Let \( Y_h = \int_a^b f \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) dx \). By (2.25) we see that

\[ \text{Var} \left( Y_h \right) = \sum_{m=k_0}^{k_1} a_{2m}^2 \int_a^b \int_a^b (\rho_h(x-y))^{2m} dx dy. \] (2.28)

Since \( |\rho_h(x-y)| \leq 1 \) we see that

\[ a_{2k_0}^2 \int_a^b \int_a^b (\rho_h(x-y))^{2k_0} dx dy \]

(2.29)

\[ \leq \text{Var} \left( Y_h \right) \leq \sum_{m=k_0}^{k_1} a_{2m}^2 \int_a^b \int_a^b (\rho_h(x-y))^{2k_0} dx dy. \]

We obtain (1.13) by showing that, in the limit, as \( h \downarrow 0 \), the moments of the left-hand side are equal to the moments of the right-hand side, (as in the proof of Lemma 2.1). We have

\[ E \left\{ \left( \sum_{m=k_0}^{k_1} a_{2m} \int_a^b H_{2m} \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) dx \right)^n \right\} \] (2.30)
\[
\sum_{m_i=k_0; i=1,\ldots, n}^{k_1} \left( \prod_{i=1}^{n} a_{2m_i} \right) \\
\int_{[a,b]^n} E \left\{ \prod_{i=1}^{n} H_{2m_i} \left( \frac{G(x_i + h) - G(x_i)}{\sigma(h)} \right) \right\} \prod_{i=1}^{n} dx_i.
\]

As in the proof of Theorem 2.1 we have

\[
E \left\{ \prod_{i=1}^{n} H_{2m_i} \left( \frac{G(x_i + h) - G(x_i)}{\sigma(h)} \right) \right\} = \sum_{l=1}^{[n/2]} \sum_{C_1 \cup C_2 \cup \cdots \cup C_l = \{x_i, i=1,\ldots, n\}}^{l} \prod_{i=1}^{l} \sum_{\pi \in \mathcal{P}(C_i) \cup \{i,i'\}\in\pi} \left( \prod_{i,i'} \rho_h(x_i - x_{i'}) \right)
\]

where the second sum runs over all partitions of \( \{x_i, i = 1,\ldots, n\} \) into \( l \) sets, \( C_1,\ldots, C_l \) with \( |C_i| \geq 2, i = 1,\ldots, l \). \( (|C| := \# \text{ of elements in } C) \) and if \( C = \{x_1,\ldots, x_k\} \), then \( \mathcal{P}(C) \) is the set of pairings of the \( \sum_{i=1}^{k} 2m_i \) elements consisting of \( 2m_i \) copies of the letter \( x_i \) subject to the same two restrictions as in the proof of Lemma 2.1:

(i) no single letter \( x_i \) is paired with itself;

(ii) for any partition \( C = A \cup B \), at least one letter of \( A \), is paired with a letter of \( B \).

Of course all \( k_0 \leq m_i \leq k_1 \).

Let

\[
\mathcal{G} = \{ C_1 \cup C_2 \cup \cdots \cup C_l = \{x_i, i = 1,\ldots, n\} | |C_i| = 2, i = 1,\ldots, l \}. \tag{2.32}
\]

Then necessarily for partitions in \( \mathcal{G} \), \( n \) is even, \( l = n/2 \) and the restrictions on \( \mathcal{P}(C_i) \) show that if \( C_i = \{x_i, x_j\} \) then \( m_{2j} = m_{2i} \). In this case the contribution to the last line of (2.30) is

\[
\prod_{i=1}^{n/2} \left( \int_a^b \left( \int_a^b (\rho_h(x - y))^2 \right)^{2m_i} dx dy \right). \tag{2.33}
\]

There are \( \frac{(2l)!}{2^l l!} \) pairings of \( \{x_i, i = 1,\ldots, n = 2l\} \). Hence the contribution of all the partitions in \( \mathcal{G} \) to (2.30) is

\[
\frac{(2l)!}{2^l l!} \sum_{m_i=k_0; i=1,\ldots,n/2}^{k_1} \left( \prod_{i=1}^{n/2} a_{2m_i}^2 \left( \int_a^b \left( \int_a^b (\rho_h(x - y))^2 \right)^{2m_i} dx dy \right) \right) \tag{2.34}
\]

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where the last line comes from (2.28).

Thus, as in the proof of Lemma 2.1, it suffices to show that for any set say $C_p$, with $|C_p| \geq 3$, and any $\pi \in \mathcal{P}(C_p)$

$$\int_{[a,b][C_p]} \prod_{(i,i') \in \pi} \rho_h(x_i - x_{i'}) \prod_{x_i \in C_p} dx_i = o \left( \left( \text{Var} \left( Y_h \right) \right)^{|C_p|/2} \right).$$

(2.35)

Suppose that $|C_p| = k$. We relabel the elements of $C_p$, $x_1, \ldots, x_k$ and choose them so that $m_1 \leq m_2 \leq \ldots \leq m_k$. If there are no strict inequalities, i.e., if $m_1 = m_2 = \ldots = m_k$, then, because of (1.12), we are in the same situation as in the proof of Theorem 2.1 and we obtain

$$\int_{[a,b][C_p]} \prod_{(i,i') \in \pi} \rho_h(x_i - x_{i'}) \prod_{x_i \in C_p} dx_i = o \left( \left( \text{Var} \left( Y_h \right) \right)^{|C_p|/2} \right).$$

(2.36)

Using (2.29) and the fact that $|\rho_h(\cdot)| \leq 1$, we see that this implies (2.35).

If there is at least one strict inequality, that is, if $m_j < m_{j+1}$, for at least one $1 \leq j \leq k$, it follows from the second restriction on $\pi$, that we can find some $(j,j') \in \pi$ with $m_j < m_{j'}$. Set

$$\|\rho_h\|_{2m_i} = \left( \int_a^b \int_a^b (\rho_h(x - y))^{2m_i} dx dy \right)^{1/2m_i}.$$  

(2.37)

Using (2.29) again and the fact that $\|\rho_h\|_{2m_i} \leq \|\rho_h\|_{2k_0}$, we see that to obtain (2.35), it suffices to show that

$$\int_{[a,b][C_p]} \prod_{(i,i') \in \pi} \rho_h(x_i - x_{i'}) \prod_{x_i \in C_p} dx_i = o \left( \prod_{i=1}^k \|\rho_h\|_{2m_i}^{m_i} \right).$$

(2.38)

To show that (2.38) holds, we successively bound the integrals over $x_1, x_2, \ldots, x_k$ using Hölder’s inequality, as described in the proof of Theorem 2.1. This shows that each factor of the form $\rho_h(x_i - x_{i'})$ with $i < i'$ makes a contribution which is $O(\|\rho_h\|_{2m_i})$. When $m_i = m_{i'}$ we can write this as
\[ O \left( \| \rho_h \|_{2m_i}^{1/2} \| \rho_h \|_{2m_i'}^{1/2} \right). \] When \( m_i < m_i' \) it follows from (1.12) that the bound
\[ O \left( \| \rho_h \|_{2m_i}^{1/2} \| \rho_h \|_{2m_i'}^{1/2} \right) = O \left( \| \rho_h \|_{2m_i}^{1/2} \| \rho_h \|_{2m_i'}^{1/2} \right). \]

Since there are \( 2m_i \) factors containing \( x_i \) for each \( i \), we always get a bound which is \( O \left( \prod_{i=1}^{k} \| \rho_h \|_{2m_i}^{m_i} \right). \) The desired estimate (2.38) follows because, as we pointed out above, for some \((j, j') \in \pi, m_j < m_j' \). Thus we get (1.13) when the Hermite polynomial expansion of \( f \) contains a finite number of terms.

To remove this restriction consider an \( f \) as in (2.26) and let
\[ f_n(x) = \sum_{m=k_0}^{n} a_{2m} H_{2m}(x) \quad (2.39) \]
Set \( Y_h = \int_a^b (G(x+y) - G(x)) \, dx \) and \( Y_{n,h} = \int_a^b f_n (G(x+y) - G(x)) \, dx \). Using (2.25) and the fact that \( \int_a^b \rho_h(x-y) 2^m \, dx \, dy \) is decreasing as \( m \) increases, we have
\[ \lim_{n \to \infty} \sup_{h} E \left\{ \left( \frac{Y_h - Y_{n,h}}{\sqrt{\operatorname{Var}(Y_h)}} \right)^2 \right\} = \lim_{n \to \infty} \sup_{h} \frac{\sum_{m=n+1}^{\infty} a_{2m}^2 \int_a^b \int_a^b (\rho_h(x-y)) 2^m \, dx \, dy}{\sum_{m=k_0}^{\infty} a_{2m}^2 \int_a^b \int_a^b (\rho_h(x-y)) 2^m \, dx \, dy} \leq \lim_{n \to \infty} \sup_{h} \frac{1}{a_{2k_0}^2} \sum_{m=n+1}^{\infty} a_{2m}^2 = 0. \]

Therefore, we can take the weak limit of
\[ \lim_{h \to 0} \frac{I_G(f_n, h; a, b) - (b-a) Ef_n(\eta)}{\sqrt{\operatorname{Var}(I_G(f_n, h; a, b))}} \quad (2.41) \]
as \( n \to \infty \) and obtain (1.13).

We get the following simple corollary of Theorem 2.1 in which gives a weaker condition than (1.12) when an additional regularity condition is satisfied.
Corollary 2.1 Let $f \in L^2(R^1, d\mu)$ be symmetric and suppose that its Hermite polynomial expansion is such that (1.9) holds. Assume that (1.11) holds for all $j \in N$. Assume, furthermore, that for all $1 \leq j < 2k_0$

$$\left(\int_a^b |\rho_h(x-y)|^j \, dx \, dy\right)^{1/j} = o\left(\int_a^b \int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy\right)^{1/(2k_0)}$$

and

$$\liminf_{h \downarrow 0} \frac{\int_a^b |\rho_h(x-y)|^{2k_0+2} \, dx \, dy}{\int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy} > 0.$$  

Then

$$\lim_{h \downarrow 0} \frac{I_G(f, h; a, b) - (b-a)Ef(\eta)}{\sqrt{\text{Var} I_G(f, h; a, b)}} \xrightarrow{law} N(0,1).$$

Proof We write

$$|\rho_h(x-y)|^{2k_0+1} = |\rho_h(x-y)|^{k_0} |\rho_h(x-y)|^{k_0+1}$$

and use the Schwarz Inequality to see that

$$\frac{\int_a^b |\rho_h(x-y)|^{2k_0+2} \, dx \, dy}{\int_a^b |\rho_h(x-y)|^{2k_0+1} \, dx \, dy} \geq \frac{\int_a^b |\rho_h(x-y)|^{2k_0+1} \, dx \, dy}{\int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy}.$$  

(2.46)

It follows from (2.43) that there exists a $\delta > 0$ for which

$$\liminf_{h \downarrow 0} \frac{\int_a^b |\rho_h(x-y)|^{2k_0+1} \, dx \, dy}{\int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy} = \delta.$$  

(2.47)

Consequently, for all $l > 2k_0$

$$\liminf_{h \downarrow 0} \frac{\int_a^b |\rho_h(x-y)|^l \, dx \, dy}{\int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy} \geq \delta^{l-2k_0}.$$  

(2.48)

This shows that all the integrals $\int_a^b |\rho_h(x-y)|^j \, dx \, dy$ with $2k_0 \leq j$ have the same order of magnitude as $h$ decreases to zero. Therefore,

$$\left(\int_a^b |\rho_h(x-y)|^j \, dx \, dy\right)^{1/j} = o\left(\int_a^b \int_a^b |\rho_h(x-y)|^{j+1} \, dx \, dy\right)^{1/(j+1)}$$

(2.49)
for all $2k_0 \leq j$. This and (2.42) are all that is used in the proof of Theorem 2.1.

Lemma 2.1 is stated for the $2k$-th Wick power. It could just as well have been stated for the $2k$-th Hermite polynomial. As such it gives just one term in the Hermite polynomial expansion of $f \in L^2(R^1, d\mu)$. However, in some cases, depending on $\sigma^2(h)$, this suffices to give the CLT for all $f$, as we show in the next lemma.

**Lemma 2.2** Let $f \in L^2(R^1, d\mu)$ be symmetric and let

$$k_0 = \inf_{m \geq 1} \{m|a_{2m} \neq 0\}$$

for $a_{2m}$ as given in (1.7). Suppose that

$$\lim_{h \to 0} \left( \int_a^b \int_a^b |\rho_h(x-y)|^{2k_0+2} \, dx \, dy \right) = 0.$$  

Then

$$\text{Var} \, IG(f; h; a, b) \sim a_{2k_0}^2 \int_a^b \int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy$$

and

$$\lim_{h \to 0} \frac{IG(f; h; a, b) - (b-a)Ef(\eta)}{\sqrt{\text{Var} \, IG(f; h; a, b)}} \overset{\text{law}}{=} N(0, 1).$$

**Proof** It follows from (1.6), and (2.50), that

$$\int_a^b f \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) \, dx - (b-a)Ef(\eta)$$

$$= a_{2k_0} \int_a^b H_{2k_0} \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) \, dx$$

$$+ \int_a^b \left( \sum_{m=k_0+1}^{\infty} a_{2m} H_{2m} \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) \right) \, dx$$

$$:= a_{2k_0} \int_a^b H_{2k_0} \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) \, dx + \int_a^b W(x) \, dx.$$
By (2.24) and (2.25)
\[
\text{Var} \left( \int_{a}^{b} W_h(x) \, dx \right) = \sum_{m=k_0+1}^{\infty} a_{2m}^2 \int_{a}^{b} \int_{a}^{b} |\rho_h(x-y)|^{2m} \, dx \, dy
\]
(2.55)
\[
\leq \int_{a}^{b} \int_{a}^{b} |\rho_h(x-y)|^{2(k_0+1)} \, dx \, dy \left( \sum_{m=k_0+1}^{\infty} a_{2m}^2 \right).
\]

By (2.51)
\[
\text{Var} \left( \int_{a}^{b} W_h(x) \, dx \right) = o \left( \text{Var} \, I_G(f,h;a,b) \right)
\]
(2.56)
and (2.52) follows.

By (2.54)
\[
\lim_{h \to 0} \frac{\int_{a}^{b} f \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) \, dx - (b-a) \, Ef(\eta)}{\sqrt{\text{Var} \, I_G(f,h;a,b)}}
\]
(2.57)
\[
= \lim_{h \to 0} \frac{\int_{a}^{b} H_{2k_0} \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) \, dx}{\sqrt{\text{Var} \, I_G(f,h;a,b)}}
\]
\[
+ \lim_{h \to 0} \frac{\int_{a}^{b} W_h(x) \, dx}{\sqrt{\text{Var} \, I_G(f,h;a,b)}}.
\]

Using (2.56) we see that
\[
\lim_{h \to 0} \text{Var} \left( \frac{\int_{a}^{b} W_h(x) \, dx}{\sqrt{\text{Var} \, I_G(f,h;a,b)}} \right) = 0.
\]
(2.58)

Therefore, (2.53) follows from (2.57), (2.52) and (2.6).

\[\square\]

**Remark 2.1** It is easy to see that when (2.51) holds
\[
\text{Var} \, I_G(f,h;a,b) \sim (E \, (f(\eta) H_{2k_0}(\eta)))^2 \int_{a}^{b} \int_{a}^{b} |\rho_h(x-y)|^{2k_0} \, dx \, dy
\]
(2.59)
and when (2.43) holds
\[
\text{Var} \, I_G(f,h;a,b) \sim \sum_{m=k_0}^{\infty} (E \, (f(\eta) H_{2m}(\eta)))^2 \int_{a}^{b} \int_{a}^{b} |\rho_h(x-y)|^{2m} \, dx \, dy.
\]
(2.60)
3 Concave $\sigma^2$

Using the fact that $\rho_h$ is symmetric and setting $c = b - a$ we see that for all $k \in \mathbb{N}$

$$\int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy = \int_0^c \int_0^c |\rho_h(x-y)|^k \, dx \, dy = 2 \int_0^c |\rho_h(s)|^k (c-s) \, ds. \quad (3.1)$$

The function $\sigma^2(h)$, defined in (1.1), has the properties that $\sigma^2(0) = 0$, and $\sigma^2(h) \neq 0$. Therefore, if it is concave, it is also both increasing and strictly increasing on $[0, c_0]$, for some $c_0 > 0$. In what follows we assume that $c = b - a \geq c_0$.

**Lemma 3.1** When $\sigma^2(h)$ is concave on $[0, c]$, for all $0 < h << c$, and $k \in \mathbb{N}$

$$\frac{(c-h)}{2^k} \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds \leq \int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy \quad (3.2)$$

$$\leq 6c \left(1 + \frac{1}{2^k}\right) \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds.$$

The proof of Lemma 3.1 uses the next lemma which is also used to give many other properties of the integrals in (3.1).

**Lemma 3.2** When $\sigma^2(h)$ is concave, for all $0 < h << c$, and $k \in \mathbb{N}$

$$\frac{1}{2^{k+1}} \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds \leq \int_0^c |\phi_h(s)|^k \, ds \quad (3.3)$$

$$\leq 3 \left(1 + \frac{1}{2^k}\right) \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds.$$

**Proof** It is useful to work with $-\phi_h(s)$ rather than $\phi_h(s)$. To avoid confusion we set $\varphi_h(s) = -\phi_h(s)$. Obviously $|\varphi_h(s)| = |\phi_h(s)|$. Using the fact that $\sigma^2(s)$ is concave, we note that for $0 < s \leq h$,

$$\varphi_h(s) = \frac{1}{2} (\sigma^2(h+s) + \sigma^2(h-s) - 2\sigma^2(s)) \quad (3.4)$$

$$\leq \left(\sigma^2(h) - \sigma^2(s)\right).$$

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Since $\sigma^2(s)$ is increasing, by writing

$$\varphi_h(s) = \frac{1}{2}(\sigma^2(h - s) - \sigma^2(s)) + (\sigma^2(h + s) - \sigma^2(s))$$  \hspace{1cm} (3.5)$$

we see that

$$\varphi_h(s) \geq 0 \quad \text{for} \quad s \in [0, h/2].$$  \hspace{1cm} (3.6)$$

Let

$$A_h := \{0 < s \leq h \mid \varphi_h(s) < 0\}.$$  \hspace{1cm} (3.7)$$

Clearly $A_h \subset (h/2, h]$. Furthermore, on $A_h$, since $\sigma^2(s)$ is increasing

$$|\varphi_h(s)| = \frac{1}{2}((\sigma^2(s) - \sigma^2(h - s)) - (\sigma^2(h + s) - \sigma^2(s))) \leq \frac{1}{2}(\sigma^2(s) - \sigma^2(h - s)).$$  \hspace{1cm} (3.8)$$

Let

$$B_h := \{0 < s \leq h \mid 0 \leq \varphi_h(s)\}.$$  \hspace{1cm} (3.9)$$

Then, by (3.4) and (3.8)

$$\int_0^h |\varphi_h(s)|^k \, ds = \int_{B_h} |\varphi_h(s)|^k \, ds + \int_{A_h} |\varphi_h(s)|^k \, ds \leq \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds + \frac{1}{2^k} \int_{h/2}^h |\sigma^2(s) - \sigma^2(h - s)|^k \, ds$$

$$= \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds + \frac{1}{2^k} \int_0^{h/2} |\sigma^2(s + h/2) - \sigma^2(h/2 - s)|^k \, ds.$$  \hspace{1cm} (3.10)$$

Using the fact that $\sigma^2(s)$ is monotonically increasing, when $0 \leq s \leq h/2$, we have $0 \leq \sigma^2(s + h/2) - \sigma^2(h/2 - s) \leq \sigma^2(h) - \sigma^2(h/2 - s)$. Consequently,

$$\int_0^{h/2} |\sigma^2(s + h/2) - \sigma^2(h/2 - s)|^k \, ds \leq \int_0^{h/2} |\sigma^2(h) - \sigma^2(h/2 - s)|^k \, ds$$

$$= \int_0^{h/2} |\sigma^2(s) - \sigma^2(h)|^k \, ds,$$  \hspace{1cm} (3.11)$$

where the last step employs a simple change of variables. This shows us that

$$\int_0^h |\varphi_h(s)|^k \, ds = \int_0^h |\varphi(s)|^k \, ds \leq \left(1 + \frac{1}{2^k}\right) \int_0^h |\sigma^2(h) - \sigma^2(s)|^k \, ds.$$  \hspace{1cm} (3.12)$$
Let \( g \) be a convex increasing function with \( g(0) = 0 \). Then, if \( a \geq b \geq 0 \),
\[ g((a - b)) \leq g(a) - g(b). \]
Therefore, since \( \sigma^2 \) is concave and increasing
\[
\int_h^c g(2|\phi_h(s)|) \, ds
= \int_h^c g(\sigma^2(s) - \sigma^2(s - h) - (\sigma^2(s + h) - \sigma^2(s))) \, ds
\leq \int_h^c g(\sigma^2(s) - \sigma^2(s - h)) \, ds - \int_h^c g(\sigma^2(s + h) - \sigma^2(s)) \, ds
\]
\[= \int_h^c g(\sigma^2(s) - \sigma^2(s - h)) \, ds - \int_{2h}^{c+h} g(\sigma^2(s) - \sigma^2(s - h)) \, ds \]
\[\leq \int_{2h}^{h} g(\sigma^2(s) - \sigma^2(s - h)) \, ds \]
\[= \int_0^h g(\sigma^2(s + h) - \sigma^2(s)) \, ds \]
\[\leq 2 \int_0^{h/2} g(\sigma^2(s + h) - \sigma^2(s)) \, ds. \]

On the other hand, using (3.6)
\[
\int_0^{h/2} g(2|\phi_h(s)|) \, ds
= \int_0^{h/2} g\left((\sigma^2(s + h) - \sigma^2(s)) + (\sigma^2(h - s) - \sigma^2(s))\right) \, ds
\geq \int_0^{h/2} g\left(\sigma^2(s + h) - \sigma^2(s)\right) \, ds.
\]
Consequently,
\[
\int_h^c g(2|\phi_h(s)|) \, ds \leq 2 \int_0^h g(2|\phi_h(s)|) \, ds \quad (3.16)
\]
and therefore
\[
\int_0^h g(2|\phi_h(s)|) \, ds \leq 3 \int_0^h g(2|\phi_h(s)|) \, ds. \quad (3.17)
\]
Using (3.17) and (3.13) with \( g(\cdot) = |\cdot|^k \) we get the upper bound in (3.3).

To get the lower bound in (3.3) we note that
\[
\int_0^h |2\phi_h(s)|^k \, ds \geq \int_0^{h/2} |2\phi_h(s)|^k \, ds
= \int_0^{h/2} \left((\sigma^2(h - s) - \sigma^2(s)) + (\sigma^2(h + s) - \sigma^2(s))\right)^k \, ds
\]
\[ \int_0^{h/2} |\sigma^2(h + s) - \sigma^2(s)|^k ds \geq \int_0^{h/2} |\sigma^2(h) - \sigma^2(s)|^k ds \]

which, since \( \sigma^2(s) \) is increasing, implies that

\[ 2 \int_0^h |2\phi_h(s)|^k ds \geq \int_0^h |\sigma^2(h) - \sigma^2(s)|^k ds. \quad (3.19) \]

**Proof of Lemma 3.1** The upper bound in (3.2) follows immediately from Lemma 3.2 and (3.1). Also, by (3.1) and (3.19)

\[ \int_a^b \int_a^b |2\phi_h(x - y)|^k dx dy = 2 \int_0^c |2\phi_h(s)|^k (c - s) ds \quad (3.20) \]
\[ \geq 2(c - h) \int_0^h |2\phi_h(s)|^k ds \]
\[ \geq (c - h) \int_0^h |\sigma^2(h) - \sigma^2(s)|^k ds. \]

This gives the lower bound in (3.2). \( \square \)

It is useful to record the following inequalities:

**Lemma 3.3** When \( \sigma^2(s) \) is concave on \([0, c]\) it follows that for some \( 0 < h << c \), and \( k \in N \)

\[ \frac{1}{2c} \int_a^b \int_a^b |\rho_h(x - y)|^k dx dy \leq \sup_{a \leq x \leq y \leq b} \int_a^b |\rho_h(x - y)|^k dy \leq \frac{3}{c - h} \int_a^b \int_a^b |\rho_h(x - y)|^k dx dy. \quad (3.21) \]

In particular (1.18) holds if and only if

\[ \lim_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x - y)|^{2k} dx dy}{\int_a^b \int_a^b |\rho_h(x - y)|^k dx dy} > 0. \quad (3.22) \]

**Proof** For all \( k \in N \)

\[ \sup_{a \leq x \leq y \leq b} \int_a^b |\rho_h(x - y)|^k dy = \sup_{a \leq x \leq b-x} \int_a^b |\rho_h(y - x)|^k dy \quad (3.23) \]
\[ = \sup_{a \leq s \leq b-x} \int_a^b |\rho_h(s)|^k ds. \]
Using this and the fact that \( \rho_h(s) = \rho_h(-s) \) we see that
\[
\int_0^c |\rho_h(s)|^k \, ds \leq \sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k \, dy \leq 2 \int_0^c |\rho_h(s)|^k \, ds. \tag{3.24}
\]

Using (3.1) and (3.24) we see that
\[
\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k \, dy \geq \frac{1}{c} \int_0^c |\rho_h(s)|^k(c - s) \, ds \tag{3.25}
\]
\[
= \frac{1}{2c} \int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy.
\]

This gives the first inequality is given in (3.21). For the second inequality we see that by (3.24), (3.17) and (3.1)
\[
\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k \, dy \leq 2 \int_0^c |\rho_h(s)|^k \, ds \tag{3.26}
\]
\[
\leq \frac{6}{c - h} \int_0^h |\rho_h(s)|^k(c - s) \, ds
\]
\[
\leq \frac{6}{c - h} \int_0^c |\rho_h(s)|^k(c - s) \, ds
\]
\[
= \frac{3}{c - h} \int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy.
\]

The rest of the lemma is obvious. \( \square \)

**Lemma 3.4**  When \( \sigma^2(h) \) is concave on \([0, c] \), (1.11) and (1.17) hold for all \( k \in \mathbb{N} \). In addition
\[
\left( \int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy \right)^{1/k} = o \left( \int_a^b \int_a^b |\rho_h(x - y)|^{k+1} \, dx \, dy \right)^{1/(k+1)} \tag{3.27}
\]
for all \( k \in \mathbb{N} \), so (1.12) also holds.

**Proof**  Using (3.24) and (3.17) we see that
\[
\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k \, dy \leq 2 \int_0^c |\rho_h(s)|^k \, ds \tag{3.28}
\]
\[
\leq 6 \int_0^h |\rho_h(s)|^k \, ds,
\]
which gives (1.17). (Here we use the simple observation that if
\[ \int_{c}^{h} |\phi_{h}(s)|^{k} \, ds \leq 2 \int_{0}^{h} |\phi_{h}(s)|^{k} \, ds \] (3.29)
then
\[ \int_{c}^{h} |\rho_{h}(s)|^{k} \, ds \leq 2 \int_{0}^{h} |\rho_{h}(s)|^{k} \, ds. \] (3.30)
We continue to pass between relations for \( \phi \) and \( \rho \) in this way without further
comment.)

The condition in (1.11) follows from (3.21).

To obtain (3.27) we note that for \( k < m \in \mathbb{N} \), by (3.2) used twice
\[
\left( \int_{a}^{b} \int_{a}^{b} |\phi_{h}(x, y)|^{k} \, dx \, dy \right)^{1/k} 
\leq C_{1} \left( \int_{0}^{h} |\sigma^{2}(h) - \sigma^{2}(s)|^{k} \, ds \right)^{1/k} 
\leq C_{1} h^{1/k-1/m} \left( \int_{0}^{h} |\sigma^{2}(h) - \sigma^{2}(s)|^{m} \, ds \right)^{1/m} 
\leq C_{1} h^{1/k-1/m} \left( C_{2} \int_{a}^{b} \int_{a}^{b} |\phi_{h}(x, y)|^{m} \, dx \, dy \right)^{1/m},
\] (3.31)
where \( C_{1} \) and \( C_{2} \) are finite constants that only depend on \( c = b - a \) for all \( h << c \).

**Lemma 3.5** Let \( \sigma^{2}(h) \) be concave and regularly varying with index \( \gamma > 0 \).
Then for all \( k \in \mathbb{N} \)
\[ \int_{a}^{b} \int_{a}^{b} |\rho_{h}(x - y)|^{k} \, dx \, dy \approx h. \] (3.32)

**Proof** It is clear from (3.2) that
\[ \int_{a}^{b} \int_{a}^{b} |\rho_{h}(x - y)|^{k} \, dx \, dy \leq 6c \left( 1 + \frac{1}{2^{k}} \right) h. \] (3.33)
Also by Lemma 3.1
\[
\int_{a}^{b} \int_{a}^{b} |\rho_{h}(x - y)| \, dx \, dy \geq \frac{c - h}{2} \int_{0}^{h} \left| 1 - \frac{\sigma^{2}(s)}{\sigma^{2}(h)} \right| \, ds 
\geq \frac{c - h}{2} \int_{h/2}^{h} \left| 1 - \frac{\sigma^{2}(h/2)}{\sigma^{2}(h)} \right| \, ds.
\]
When $\sigma^2(h)$ is regularly varying at zero with index $\gamma > 0$

$$
\lim_{h \to 0} \left| 1 - \frac{\sigma^2(h/2)}{\sigma^2(h)} \right| = 1 - \frac{1}{2^\gamma}.
$$

(3.35)

Using this in (3.34) we get the lower bound in (3.32).

In preparation for the next lemma we point out that when $\sigma^2(s)$ is concave and regularly varying with index $\gamma \geq 0$

$$
\lim_{s \to 0} s \frac{d}{ds} (\sigma^2(s)) = \gamma.
$$

(3.36)

This follows from the Monotone Density Theorem [1, Theorem1.7.2b], (see also [3, page 596]), since the derivative of $\sigma^2(s)$ is decreasing.

Lemma 3.6 Let $\sigma^2(s)$ be concave on $[0, c]$.

(1) If $s \frac{d}{ds} \sigma^2(s)$ is increasing on $[0, h]$, then for some $0 < h << c$ and all $k \geq 1$,

$$
\int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy \leq C_{c,k} \left( \frac{h \frac{d}{dh} \sigma(h)}{\sigma(h)} \right)^k h,
$$

where $C_{c,k} < \infty$ depends only on $c$ and $k$.

(2) If $\sigma^2(s)$ is slowly varying at zero then for all $k \geq 1$,

$$
\lim_{h \to 0} \int_a^b \int_a^b \frac{|\rho_h(x-y)|^k \, dx \, dy}{h} = 0,
$$

and

$$
\lim_{h \to 0} \int_a^b \int_a^b \frac{|\rho_h(x-y)|^k \, dx \, dy}{h} = 0.
$$

(3.38)

(3.39)

Proof By Lemma 3.1

$$
\int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy \leq 6c \left( 1 + \frac{1}{2^k} \right) \int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^k ds.
$$

(3.40)

Using integration by parts we see that

$$
\int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^k ds = \frac{k}{\sigma^2(h)} \int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^{k-1} s \frac{d}{ds} \left( \sigma^2(s) \right) ds
$$

(3.41)

$$
\leq \frac{kh}{\sigma^2(h)} \frac{d}{dh} \left( \sigma^2(h) \right) \int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^{k-1} ds,
$$
where at the last step we use the fact that \( s \frac{d}{ds} \sigma^2(s) \) is increasing on \([0, h]\). Since the last integral in (3.41) is equal to \( h \) when \( k = 1 \) we get (3.37).

To obtain (3.38) we use the first line of (3.41) to get

\[
\int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^k ds = \frac{k}{\sigma^2(h)} \int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^{k-1} s \frac{d}{ds} \left( \sigma^2(s) \right) ds
\]

(3.42)

\[
\leq k \int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^{k-1} \frac{d}{ds} \left( \sigma^2(s) \right) ds.
\]

Consequently, it follows from (3.36), with \( \gamma = 0 \), that

\[
\limsup_{h \downarrow 0} \frac{\int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^k ds}{\int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right|^{k-1} ds} = 0.
\]

(3.43)

Iterating this and using (3.40) we get (3.38).

The statement in (3.39) follows from (3.43) and Lemma 3.1.

By the first line of (3.41)

\[
\int_0^h \left| 1 - \frac{\sigma^2(s)}{\sigma^2(h)} \right| ds = \frac{1}{\sigma^2(h)} \int_0^h s \frac{d}{ds} \left( \sigma^2(s) \right) ds.
\]

(3.44)

By (3.36) when \( \sigma^2(s) \) is concave and regularly varying with index \( \gamma \geq 0 \), \( s \frac{d}{ds} \left( \sigma^2(s) \right) \) is regularly varying with index \( \gamma \geq 0 \). Using this we see that

\[
\frac{1}{\sigma^2(h)} \int_0^h s \frac{d}{ds} \left( \sigma^2(s) \right) ds \sim \frac{1}{1 + \gamma} \frac{h^2 \frac{d}{dh} \left( \sigma^2(h) \right)}{\sigma^2(h)}
\]

(3.45)

\[
= \frac{2}{1 + \gamma} \frac{h^2 \frac{d}{dh} \left( \sigma(h) \right)}{\sigma(h)}.
\]

(3.46)

Therefore, it follows from (3.2)

\[
\int_a^b \int_a^b \rho_h(x-y) \, dx \, dy \approx h \frac{d}{dh} \left( \sigma(h) \right) \frac{\sigma(h)}{\sigma(h)}.
\]

(3.47)

When \( \sigma^2(h) \) is concave, the right-hand side of (3.47) goes to \( \gamma \) as \( h \) decreases to zero. For \( \gamma > 0 \), this restates a property given in Lemma 3.5. However, when \( \gamma = 0 \) this is a refinement of (3.38).
Lemma 3.7  When $\sigma^2(h)$ is concave on $[0, c]$ and regularly varying with index $\gamma \geq 0$ and $s \frac{d}{ds} \sigma^2(s)$ is increasing for some $0 < h << c$ and all $k \geq N$.

$$\int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy \approx \left( \frac{h \frac{d}{dh} \sigma(h)}{\sigma(h)} \right)^k h. \quad (3.48)$$

**Proof**  By (3.31)

$$C_1 h^{1-1/k} \left( \int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy \right)^{1/k} \geq \int_a^b \int_a^b |\rho_h(x - y)| \, dx \, dy \quad (3.49)$$

or, equivalently

$$\int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy \geq C_1^{-k} \left( \int_a^b \int_a^b |\rho_h(x - y)| \, dx \, dy \right)^k h. \quad (3.50)$$

where $C_1 > 0$ depends only on $c = b - a$ for all $h << c$. Using this, (3.47) and (3.37) completes the proof.

It follows from Lemma 3.3 that when

$$\lim_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x, y)|^k \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x, y)| \, dx \, dy} = 0 \quad (3.51)$$

the condition in (1.18) fails. Therefore, by the first line of (3.41) and Lemma 3.1, if $\sigma^2(h)$ is concave and

$$\lim_{h \downarrow 0} h \frac{d}{dh} \left( \log \sigma^2(h) \right) = \lim_{h \downarrow 0} \frac{h}{\sigma^2(h)} \frac{d}{dh} \left( \sigma^2(h) \right) = 0 \quad (3.52)$$

(1.18) fails.

Lemma 3.8  Assume that $\sigma^2(s)$ is concave on $[0, h]$ for some $0 < h << c$. Write

$$\sigma^2(s) = \exp \left( f(\log 1/s) \right) \quad (3.53)$$

If $\lim_{x \to \infty} f'(x) = 0$, (1.18) fails.

**Proof**  This is simple since

$$s \frac{d}{ds} \left( \log \sigma^2(s) \right) = s \frac{d}{ds} \left( f(\log 1/s) \right) = -f'(\log 1/s). \quad (3.54)$$

The assertion follows from (3.52).
4 $\sigma^2(h) = h^r$, $0 < r \leq 2$

In these cases we can find precise asymptotic limits at zero of the double integral in (3.2) and thus obtain a precise value for $\text{Var } I_G(f, h; a, b)$. We begin with the following estimates:

**Lemma 4.1** Let $\sigma^2(h) = h^r$, $0 < r \leq 2$.

When $(2 - r)k < 1$

$$\int_0^c |\phi_h(s)|^k \, ds \sim \frac{r^k|r - 1|^k c^{(r - 2)k + 1}}{2^k(r - 2)k + 1} h^{2k} \quad \text{at zero.} \quad (4.1)$$

When $r = 1$

$$\int_0^c |\phi_h(s)|^k \, ds = \frac{h^{k+1}}{k+1}. \quad (4.2)$$

When $(2 - r)k = 1$, $k \geq 2$

$$\int_0^c |\phi_h(s)|^k \, ds \sim \left|\frac{r(r - 1)}{2}\right|^k h^{2k} \log 1/h \quad \text{at zero.} \quad (4.3)$$

If $(2 - r)k > 1$

$$\int_0^c |\phi_h(s)|^k \, ds \sim h^{rk+1} \int_0^\infty |\phi_1(s)|^k \, ds \quad \text{at zero.} \quad (4.4)$$

**Proof** The equality in (4.2) is a trivial direct computation. We proceed to the others. By a simple change of variables we have

$$\int_0^c |2\phi_h(s)|^k \, ds \quad (4.5)$$

$$= \int_0^c \left|s + h|^r + |s - h|^r - 2|s|^r \right|^k \, ds$$

$$= h^{rk+1} \int_0^{c/h} \left|s + 1|^r + |s - 1|^r - 2|s|^r \right|^k \, ds$$

$$= h^{rk+1} \int_0^{c/h} |2\phi_1(s)|^k \, ds.$$
The first integral is a finite number. For the second integral we have, for $h < c/2$

$$\int_2^{c/h} \left| s + 1 \right|^r + \left| s - 1 \right|^r - 2 \left| s \right|^r \right|^k ds$$

(4.7)

$$= \int_2^{c/h} s^k \left| 1 + s^{-1} \right|^r + \left| 1 - s^{-1} \right|^r - 2 \right|^k ds$$

$$= \int_2^{c/h} s^k \left| r(r-1)s^{-2} + O(s^{-3}) \right|^k ds$$

$$= \int_2^{c/h} s^{(r-2)}(r-1) + O(s^{-1}) \right|^k ds.$$  

Using (4.5)–(4.7) we get (4.1), (4.3) and (4.4). For (4.1) and (4.3), to do the integration, it is helpful to note that when $(2-r)k < 1$, $(r-2)k > -1$ and, obviously, when $(2-r)k = 1$, $(r-2)k = -1$. For (4.4) we have $(r-2)k < -1$ so that the last integral in (4.7), and hence in (4.6), is finite.

We now consider the integral in (3.1).

**Lemma 4.2** Let $\sigma^2(h) = h^r$, $0 < r \leq 2$ and set $c = b - a$.

When $(2-r)k < 1$

$$\int_a^b \int_a^b |\phi_h(x-y)|^k dx dy \sim \frac{2r^k |r-1|^{k} c^{(r-2)k+2}}{2^k((r-2)k + 1)((r-2)k + 2)} h^{2k}$$

at zero.  

(4.8)

When $r = 1$

$$\int_a^b \int_a^b |\phi_h(x-y)|^k dx dy \sim 2c \frac{h^{k+1}}{k+1}$$

at zero.  

(4.9)

When $(2-r)k = 1$, $k \geq 2$

$$\int_a^b \int_a^b |\phi_h(x-y)|^k dx dy \sim 2c \frac{h^{2k} \log 1/h}{2}$$

at zero.  

(4.10)

When $(2-r)k > 1$

$$\int_a^b \int_a^b |\phi_h(x-y)|^k dx dy \sim 2ch^{r^k+1} \int_0^\infty |\phi_1(s)|^k ds$$

at zero.  

(4.11)
Proof  By (3.1) it suffices to consider
\[
2 \int_0^c |\phi_h(s)|^k (c - s) \, ds = 2c \int_0^c |\phi_h(s)|^k \, ds - 2 \int_0^c |\phi_h(s)|^k s \, ds. \tag{4.12}
\]
As in the proof of Lemma 4.1, (4.9) is a trivial direct computation. We consider the others. The first integral on the right-hand side of (4.12) is handled by Lemma 4.1 and obviously gives the results in Lemma 4.1 multiplied by 2c. For the last integral, by a change of variables, we have
\[
\int_0^c |2\phi_h(s)|^k s \, ds \tag{4.13}
\]
\[
= \int_0^c \left| |s + h|^r + |s - h|^r - 2|s|^r \right|^k s \, ds
\]
\[
= h^{rk+2} \int_0^{c/h} \left| |s + 1|^r + |s - 1|^r - 2|s|^r \right|^k s \, ds
\]
\[
= h^{rk+2} \int_0^{c/h} |2\phi_1(s)|^k s \, ds.
\]
In the case of (4.11) as in (4.7) we can bound (4.13) by
\[
Ch^{rk+2} \int_{2}^{c/h} \frac{1}{s^{(2-r)k-1}} \, ds. \tag{4.14}
\]
If \((2 - r)k > 2\) the integral is bounded whereas if \((2 - r)k = 2\) the integral \(\approx \log 1/h\). Thus the last integral in (4.13) contributes nothing to the asymptotic estimate of (4.12) at zero in these cases. When \(1 < (2 - r)k < 2\) we see that (4.14) is equal to \(Ch^{rk+2}h^{(2-r)k-2} = Ch^{rk+(2-r)k} = o(h^{rk+1})\) since \(1 < (2 - r)k\). Hence the last integral in (4.13) contributes nothing to the asymptotic estimate of (4.12) at zero in this case as well.

In the cases of (4.8) and (4.10) we compute the integral in (4.13) using (4.6) and (4.7). We see that it contributes nothing to the asymptotic estimate at zero in (4.10) but it does enter into the estimates in (4.8). \(\Box\)

We write the estimates in Lemma 4.2 in different forms that are useful to us.

Corollary 4.1 Let \(\sigma^2(h) = h^r, \, 0 < r \leq 2\).
\[
\left( \int_a^b \int_a^b |\phi_h(x - y)|^k \, dx \, dy \right)^{1/k} \sim \begin{cases} 
D_{1,k} h^{2(2-r)k+1} & \text{if } 2-r < 1 \\
\left( \frac{2c}{k+1} \right)^{1/k} h^{1+1/k} & \text{if } r = 1 \\
D_{2,k} h^{2} \left( \frac{2}{k+1} \right)^{1/k} h^{1+1/k} & \text{if } 2-r = 1, \, k \geq 2 \\
D_{3,k} h^{r+1/k} & \text{if } 2-r > 1
\end{cases}
\]
Also
\[
\int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy \sim \begin{cases} 
D_{4,k} h^{(2-r)k} & (2-r)k < 1 \\
\frac{2c}{k+1} h & r = 1 \\
D_{5,k} h (\log 1/h) & (2-r)k = 1, k \geq 2 \\
D_{6,k} h & (2-r)k > 1
\end{cases}
\]

Here \( D_{j,k} = D_{j,k}(r,c), j = 1, \ldots, 6, \) do not depend on \( h. \) (They can be obtained from Lemma 4.2.)

5 Proofs of Theorems 1.1 and 1.2 and Tables 1 and 2

Proof of Theorem 1.1 All we need to do is verify that the hypotheses of Theorem 1.3 are satisfied. When \( \sigma^2(h) \) is concave we show this in Lemma 3.4. It remains to consider \( \sigma^2(h) = h^r, 1 < r \leq 3/2. \) As we show in (3.24)
\[
\sup_{a \leq x \leq b} \int_a^b |\rho_h(x-y)|^k \, dy \leq 2 \int_0^c |\rho_h(s)|^k \, ds. \tag{5.1}
\]
and, as we show in (3.1)
\[
\int_a^b \int_a^b |\rho_h(x-y)|^k \, dx \, dy = 2 \int_0^c |\rho_h(s)|^k (c-s) \, ds. \tag{5.2}
\]
One can see from Lemmas 4.1 and 4.2 that the right-hand sides of (5.1) and (5.2) have the same asymptotic behavior at zero for all \( \sigma^2(h) = h^r, 0 < r < 2. \) Thus we have (1.11) when \( \sigma^2(h) = h^r, 1 < r < 2. \)

We now show that when \( \sigma^2(h) = h^r, 1 < r \leq 3/2 \)
\[
\left( \int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \right)^{1/j} = o \left( \int_a^b \int_a^b |\rho_h(x-y)|^{2k} \, dx \, dy \right)^{1/(j+1)} \tag{5.3}
\]
for all \( j \in \mathbb{N} \) which, of course, implies (1.12). By Corollary 4.1
\[
\int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \sim D_{4,1} h^{(2-r)} \tag{5.4}
\]
and when \( j > 2 \)
\[
\int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \sim D_{6,j} h \tag{5.5}
\]
\[
\int_a^b \int_a^b |\rho_h(x-y)|^2 \, dx \, dy \sim \begin{cases} 
D_{5,2} h (\log 1/h) & r = 3/2 \\
D_{6,2} h & 1 < r < 3/2 \end{cases}.
\]

(5.6)

When \(1 < r < 3/2\), \(2 - r > 1/2\) and (5.5) holds for all \(j \geq 2\). Thus we get (5.3). When \(r = 3/2\), \(2 - r = 1/2\) but we get the extra \(\log 1/h\) term in (5.6) so we get (5.3) in this case as well.

**Proof of Theorem 1.2** We show that the hypotheses of Corollary 2.1 are satisfied. We already showed, in the proof of Theorem 1.1, that (1.11) holds for \(\sigma^2(h) = h^r\), \(1 < r < 2\) so, in particular it holds for \(3/2 < r \leq 2 - 1/(2k_0)\). Suppose \(r = 2 - 1/(2k_0)\). Then by Corollary 4.1

\[
\left( \int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \right)^{1/j} \sim (D_{4,1})^{1/j} h^{(2-r)}
\]

and

\[
\left( \int_a^b \int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy \right)^{1/(2k_0)} \sim (D_{5,2} h (\log 1/h))^{1/(2k_0)}.
\]

(5.8)

Since, in this case \(2 - r = 1/(2k_0)\) we see that (2.42) holds. Also, by Corollary 4.1, when \(j > 2k_0\), \((2-r)j > 1\), and

\[
\int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \sim D_{6,k} h.
\]

(5.9)

Thus (2.43) is also satisfied.

When \(3/2 < r < 2 - 1/(2k_0)\) it follows from Corollary 4.1 that

\[
\left( \int_a^b \int_a^b |\rho_h(x-y)|^{2k_0} \, dx \, dy \right)^{1/(2k_0)} \sim (D_{6,2} h)^{1/(2k_0)}
\]

(5.10)

and for \(j < 2k_0\)

\[
\left( \int_a^b \int_a^b |\rho_h(x-y)|^j \, dx \, dy \right)^{1/j} \sim \begin{cases} 
(D_{4,j})^{1/j} h^{(2-r)} & (2-r)j < 1 \\
(D_{5,k} h (\log 1/h))^{1/j} (2-r)j = 1 \\
(D_{6,k} h)^{1/j} (2-r)j > 1
\end{cases}
\]

Since, in this case both \(2 - r > 1/(2k_0)\) and \(1/j > 1/(2k_0)\), (2.42) holds. When \(j \geq 2k_0\) we are in the same situation as in (5.9) so (2.43) is also satisfied. \(\square\)
Explanation of how the entries in Table 1 are obtained: Entries (1) – (4) are given in Corollary 4.1. Entry (5) is given in Lemma 3.5. Entries (6) and (7) follow from Lemma 3.7. The constants in (1.15) are taken from Lemma 4.2.

Explanation of how the entries in Table 2 are obtained: As we point out just before Corollary 1.1, $k_0 = 1$, and $a_2 = E(|\eta|^p|\eta^2 - 1|)/\sqrt{2} > 0$. The variance $\Phi(h)$ is given in (1.14) and we get the asymptotic estimates for $\int_a^b \int_a^b (\rho_h(x - y))^2 dxdy$ from Table 1 for (1)-(3) and from Lemma 3.7 for (5). Recall Remark 2.1. In (1) and (5), (2.51) holds so $\Phi(h)$ is the single term $a_2^2 \int_a^b \int_a^b (\rho_h(x - y))^2 dxdy$. In (2) we get the infinite series. Example (3) is simply (2) with the integral evaluated. For (4) we see by Lemma 3.5 that (2.43) holds. Since the variance contains an infinite number of terms we also need to use (3.33) to get the estimate for $\Phi(h)$.

When $k_0 \geq 2$ and $a^2(h) = h^r$, $3/2 < r \leq 2 - 1/(2k_0)$ we can also get precise asymptotic estimates for the denominator in (1.5). We leave this to the interested reader.

Remark 5.1 The CLT in [6, Theorem 2.2], as it applies to $I_G(f,h;a,b)$, is contained in Theorem 1.3. Condition (1.18), and the fact that $|\rho_h(\cdot)| \leq 1$, implies that there exists an absolute constant $C > 0$ such that for all $k \in N$

$$\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)| dy \leq C' \int_a^b \int_a^b |\rho_h(x - y)|^k dxdy. \quad (5.11)$$

Using the fact that $|\rho_h(\cdot)| \leq 1$, once again, we see that (5.11) implies that for all $k \in N$

$$\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k dy \leq C' \int_a^b \int_a^b |\rho_h(x - y)|^k dxdy. \quad (5.12)$$

Thus (1.18) implies (1.11), with a fixed constant $C$. Since we always have

$$\int_a^b \int_a^b |\rho_h(x - y)|^k dxdy \leq (b-a) \sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k dy, \quad (5.13)$$

we see that when (1.18) holds

$$\sup_{a \leq x \leq b} \int_a^b |\rho_h(x - y)|^k dy \approx \int_a^b \int_a^b |\rho_h(x - y)|^k dxdy. \quad (5.14)$$
Hence condition (1.18) is equivalent to saying that there exists a $\delta > 0$ such that

$$\liminf_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x - y)|^2 \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x - y)| \, dx \, dy} \geq \delta.$$  \hspace{1cm} (5.15)

As in the proof of Corollary 2.1 we can write

$$|\rho_h(x - y)|^{k+1} = |\rho_h(x - y)|^{k/2} |\rho_h(x - y)|^{(k/2)+1}$$  \hspace{1cm} (5.16)

and use the Schwarz Inequality to see that

$$\frac{\int_a^b \int_a^b |\rho_h(x - y)|^{k+2} \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x - y)|^{k+1} \, dx \, dy} \geq \frac{\int_a^b \int_a^b |\rho_h(x - y)|^{k+1} \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x - y)|^k \, dx \, dy}. \hspace{1cm} (5.17)$$

It follows from (5.15) and (5.17) that, for all $j \geq 1$

$$\liminf_{h \downarrow 0} \frac{\int_a^b \int_a^b |\rho_h(x - y)|^{j+1} \, dx \, dy}{\int_a^b \int_a^b |\rho_h(x - y)|^j \, dx \, dy} \geq \delta^j. \hspace{1cm} (5.18)$$

This shows that $\int_a^b \int_a^b |\rho_h(x - y)|^j \, dx \, dy$ and $\int_a^b \int_a^b |\rho_h(x - y)|^{j+1} \, dx \, dy$ have the same order of magnitude as $h$ decreases to zero. By (1.17) and (5.14), both these integrals go to zero as $h \downarrow 0$. Therefore, (1.12) holds.

Thus we see that when the hypotheses of CLT in [6, Theorem 2.2], as it applies to $I_G(f,h; a, b)$, hold, the hypotheses of Theorem 1.3 hold.

References


