Introduction
It has been said that a picture is worth a thousand words. This is especially true in mathematics where a picture or some other type of visual model may be useful in describing a mathematical idea. Further, models have proved useful in gaining deeper understanding of an abstract concept and as a tool in problem solving. During the past twenty years, particularly due in large part to the development of visual imagery capabilities of computers, a new area of mathematics has evolved to address these visual questions. This new area of research is often referred to simply as visual mathematics.

This paper describes the broad area of visual mathematics along with the motivation for using it in teaching and learning mathematics. Further, the paper is a workbook of visual examples drawn from various levels and areas of mathematics. These examples are a combination of topics seen in high school and university courses and include ideas drawn from arithmetic, algebra, Euclidean geometry, calculus, group theory, tiling theory, fractal geometry, hyperspace, topology, and non-Euclidean geometry.

Two Sides of the Brain – Two Different Approaches to Learning
Many students progress through their academic experiences believing that they are either have a gift for understanding mathematics or that they don’t possess this gift and that mathematics is one area which may be beyond their comprehension.

While there is some basis for these preconceived beliefs about learning mathematics, there also are opportunities for understanding each individual student’s manner of learning and help them develop their own individual learning style.

Some of the strategies for learning mathematics may be attributed to the different roles the two hemispheres of our brain play in understanding information and in solving problems. A common notion is that a student who is successful in learning mathematics must be a “left-brained” thinker. This notion may be borne out if one observes the roles for each side of the brain.

<table>
<thead>
<tr>
<th>Left Brain</th>
<th>Right Brain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical</td>
<td>Artistic</td>
</tr>
<tr>
<td>Abstract</td>
<td>Concrete</td>
</tr>
<tr>
<td>Objective</td>
<td>Subjective</td>
</tr>
<tr>
<td>Logical</td>
<td>Random</td>
</tr>
<tr>
<td>Critical Thinking</td>
<td>Creative Thinking</td>
</tr>
<tr>
<td>Propositional</td>
<td>Imaginative</td>
</tr>
<tr>
<td>Speaking</td>
<td>Visual</td>
</tr>
<tr>
<td>Dominates Right Brain</td>
<td>Submissive to Left Brain</td>
</tr>
<tr>
<td>Black and White</td>
<td>Color</td>
</tr>
<tr>
<td>Looks at Particular Cases</td>
<td>Looks at Big Picture</td>
</tr>
<tr>
<td>Rational</td>
<td>Holistic</td>
</tr>
</tbody>
</table>

Table 1. Roles Played by Left Brain and Right Brain
or hemisphere of the brain that are shown in Table 1. The left hemisphere focuses on logical and objective modes of learning and it devises methods for critical thinking. The right hemisphere learns in a more random and subjective manner and is better at creative thinking. One might say the left hemisphere is mathematical while the right hemisphere is artistic.

A major goal in helping students to become more successful learners in the mathematics classroom is to help them to utilize these particular strengths of both hemispheres to their brains. There are two major obstacles to becoming balanced learners. First, the left hemisphere tends to dominate the right hemisphere and further the right side tends to be submissive to this arrangement. Second, the vast majority of curriculum has been designed for left-brain students focusing on logical thinking, memorization, and accuracy. Both of these obstacles may be addressed by consciously giving students opportunities to understand mathematical ideas in a manner which forces the use of the right hemisphere of the brain.

Since the right hemisphere appeals to the visual senses a pedagogical strategy is to develop mathematical ideas through the use of visual imagery. This overall area of teaching might be referred to as visual mathematics and may include the use of historical documents, computer graphics, modern art and architecture, geometric constructions, as well as any image that helps to understand a mathematical idea in a visual way. The following illustrations show how visual mathematics may be used in the classroom.

I. Mathematics Motivated with Historical Documents
For teachers and students in the Middle East, there are numerous historical documents, which help to bring mathematical ideas to life. By investigating mathematical problems from the history of mathematics, students are able to look at these questions with the powerful computer packages of today. This not only gives the students experience with solving problems with computer algebra systems, like MuPad, but also gives them a deeper appreciation for what their ancestors accomplished without much technology. The following example involves the 13th Century Islamic astronomer and mathematician, Nasir al-Din al-Tusi (1201 – 1274 A.D.), and shows how historical manuscripts combined with computer technology and animations may excite students in understanding the mathematics of motion and geometric constructions.

Nasir al-Din al-Tusi (Figure 1) was born on 18 February 1201 AD in Tus in present day northeastern, Iran. He studied sciences and philosophy under the guidance of Kamal al-Din Ibn Yunus.

Al-Tusi was one of the greatest scientists, mathematicians, astronomers, philosophers, physicians, and theologians of his time. He wrote a variety of treatises on subjects ranging from Algebra, Arithmetic, Trigonometry and Geometry to Logic, Metaphysics, Medicine, Ethics, and Theology. The page (Figure 2) is taken from one of al-Tusi’s manuscripts on the Geometry of Euclid and contains a proof of the
Pythagorean Theorem. This proof, complete with diagram, shows a variation of an ancient Greek proof by Euclid. From the diagram alone, one can deduce that this manuscript is describing the Pythagorean Theorem, which states, “For any right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.” Animation programs are capable of taking this diagram, often referred to as the Bride’s Chair, and bringing it to life by transforming the two smaller squares to perfectly fit into the larger one.

Al Tusi created an accurate table on planetary movements complete with a star catalogue, which he called Al-Zij-Ilkhanī. He was one of the first mathematicians to develop spherical trigonometry and wrote a popular text, Al-Zij-Ilkhanī, on ethics. Perhaps his crowning achievement was the construction of an observatory in the Azerbaijan region of northwestern Iran. The Maragheh observatory contained many instruments for viewing the stars including an azimuth quadrant invented by al-Tusi himself. At the observatory, he worked on pointing out serious flaws in Ptolemy’s version of the motions of the planets. This improvement on Ptolemy’s planetary system was achieved by resolving linear motion into the sum of two circular motions in what has come to be known as the Tusi-couple. Copernicus may have used the Tusi-couple in the 16th Century in developing his theory of planetary motion.

The page (Figure 3) from a 13th Century treatise, Tadhkira fi ilm al-Haya, by al-Tusi describes the reasoning behind the Tusi-couple. Graphics designed with the computer algebra systems, MuPad, can lead students to al-Tusi’s result. An interesting experiment is to investigate what designs result as a point on a smaller circle is followed as it rolls around inside a larger circle. Several of these designs called hypocycloids from a MuPad program are shown below. Figure 4 shows the hypotrochoide that results when point on a small circle is followed as the small circle rolls around the inside of a larger circle and the ratio of the radii is 5:1. The MuPad program for this graphic is given below.
Figure 4.
Hypotrochoide Constructed with MuPad

Table 2.
Hypotrochoide with MuPad
By changing the ratio of the radii, various number theoretic issues may be explored as the hypotrochoide changes. In particular, when arriving at the ratio 2:1, a connection to al-Tusi’s theory is made. The MuPad graphics in Table 3 show five different positions of a smaller circle rolling in a larger circle. In this case, the radius of the smaller circle is one-half the radius of the larger circle. These five patterns duplicate the diagrams in al-Tusi’s 13th Century manuscript (Figure 3) and his result of two circular motions producing linear motion is seen.

Table 3. MuPad Graphics for al-Tusi’s Paper

The graphic below (Figure 5) shows another version of al-Tusi’s description of the Tusi-couple, which shows Arabic letters along the vertical axis and at the point of intersection of the two circles. The other graphic (Figure 6) below shows a reproduction of the diagram in al-Tusi’s manuscript showing the Latin letters as they appeared in a text of Copernicus’ results from the 16th Century. It is this connection, which leads many historians to believe that Copernicus utilized the mathematical results of al-Tusi to advance his heliocentric (sun centered) theory of planetary motion.

This is but one example drawn from the life of this extraordinary man. Possibly the greatest accomplishment of al-Tusi was in bringing together a remarkable group of scientists, mathematicians, and philosophers to the Maragheh observatory. Al-Tusi’s influence has been described in the Dictionary of Scientific Biography in the following manner.

“Al-Tusi’s influence, especially in eastern Islam, was immense. Probably, if we take all fields into account, he was more responsible for the revival of the Islamic sciences
than any other individual. His bringing together so many competent scholars and scientists at Maragheh resulted not only in the revival of mathematics and astronomy but also in the renewal of Islamic philosophy and even theology”.

II. Geometry Motivated by Islamic Tilings

In the past one hundred years, a new area of mathematics has been developed to answer questions relating to the manner in which two-dimensional polygons fit together to cover the plane. This theory has evolved to include the study of ways in which three-dimensional polyhedra may be joined together to fill up three-dimensional space and has been generalized to answer these same types of questions in n-dimensional hyperspaces. Due to its historic beginnings in two-dimensions and its natural connection to actual ceramic tilings, this area of mathematics has come to be known as tiling theory. By focusing on particular tilings, students may be led into projects which involve geometric constructions and the study of group theory. Islamic art has an unparalleled history of intricate geometric design and has produced numerous tilings which are found in mosques, mausoleums and minarets throughout the world.

Below are two examples of Islamic tilings from the Alhambra Palace which overlooks the city of Granada in southern Spain. The Alhambra is a fine example of Moorish architecture constructed in the 14th Century AD and has many fine examples of Islamic art preserved from this earlier period.

The tiling shown in Figure 7 is periodic and is an excellent example of a tiling which may be reproduced in a periodic fashion. It may be used to motivate a number of ideas in mathematical symmetry including rotational, reflectional, and translational isometries. From these basic concepts, one is able to deduce what minimal piece of the pattern, called the fundamental region, is required to generate the entire pattern. Each of these periodic tilings may be constructed in this fashion by hand or by computer. Examples of these periodic tilings fall into seventeen classes of design according to the rotations and reflections that they contain.

![Figure 7: Periodic Tiling – Alhambra Palace](image1.png)

![Figure 8: Tiling Centered Around a Moroccan Star – Alhambra Palace](image2.png)

The tiling in Figure 8 is constructed around a central eight-pointed Moroccan star and generated outward and so is not periodic in the same manner as the previous example. The symmetry in this tiling motivates topics in finite group theory. Since patterns of this type contain exactly one center of rotation and all lines of reflection pass through this point then these radial patterns may be used to classify the different cyclic and
dihedral groups. The generation of this type of tiling by computer proves more challenging since it cannot be generated by a uniform fundamental region.

The periodic tilings may be generated by computer software. A tiling from the Alhambra Palace generated using the program *Geometer’s Sketchpad* is shown below in Table 4.

<table>
<thead>
<tr>
<th>Islamic Tiling from the Alhambra Palace, Granada, Spain</th>
<th>Detail of the Tiling on Right Side</th>
<th>A Fundamental Triangle Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>Creating the Fundamental Star</td>
<td>Removing the Original Triangle</td>
<td>Constructing a Star of David</td>
</tr>
<tr>
<td>Converting a Star of David</td>
<td>Constructing a Regular Hexagon</td>
<td></td>
</tr>
<tr>
<td>Combining Stars and Hexagons</td>
<td>Coloring the Fundamental Tile</td>
<td>Tiling by Translations</td>
</tr>
<tr>
<td></td>
<td>A Completed Tiling</td>
<td></td>
</tr>
</tbody>
</table>

*Table 4. Geometer’s Sketchpad Construction of an Islamic Tiling*

### III. Calculus Motivated by Computer Graphics

The concept of limit lies at the heart of calculus. An interesting example for students involves the amount of carpet that is needed to carpet stairs that span 4 meters in the horizontal direction and 3 meters in the vertical direction. The stairs are to be carpeted on all the horizontal and vertical surfaces.
It is easy to calculate that if four 4 stairs are constructed in this space then a total of 7 meters of carpet is needed to carpet the stairs. If 8, 16, 32, or even 1,000,000 stairs are constructed in the same space then theoretically; 7 meters of carpet will be needed to cover the stairs.

The intriguing part of the problem is to imagine what will happen “in the limit”, that is, what will happen if there are infinitely many stairs. In this infinite case, the stairs transform into a ramp and 5 meters of carpet is needed to cover it. The dynamic aspect of this problem for students lay in pondering how passing from the finite case to the infinite case changes the carpet needed from 7 meters to 5 meters with no cases in between.

Another interesting example involving a limit is developed using a surface called “Gabriel’s Horn”. This infinite surface is generated by looking at the portion of the hyperbola $y = 1/x$ that lies above the $x$-axis for all $x$ that are greater than or equal to one. This curve in the first quadrant is then rotated about the $x$-axis and the resulting surface is called Gabriel’s Horn (Figure 9).

Using methods of integration, it is possible to calculate that the volume of this horn is $\pi$ while the surface area is infinite. If “finite intuition” is applied this result appears to
say that it is possible to fill up this surface with a finite amount of paint but there is no possible way to paint this surface with a finite amount. The paradox is useful in helping students with the idea that much of the natural intuition that we have concerning finite situations does not necessarily extend to infinite circumstances.

Figure 9. Paradox of Gabriel’s Horn – Infinite Surface with Finite Volume

MuPad has a feature that investigates the concept of the definite integral by viewing the area under a curve as the limit of Riemann Sums.

IV. Topology Motivated with Computer Imagery

In recent years, many hidden worlds of mathematics have been revealed by computers with the capacity to create detailed visual images. Computer graphics in areas of mathematics like topology, tiling theory, fractal geometry, and tiling theory often are a student’s first introduction to an abstract concept. The image in Figure 10 is a computer Ilfochrome print of a Klein bottle by Tom Banchoff and Ying Wang of Brown University, USA. A Klein bottle is an unusual surface which has no inside and no outside. For this reason, this surface is referred to as non-orientable.

The image in Figure 11 shows a two-dimensional Penrose tiling on the floor which was generated by Eugenio Durand using QuasiTiler at the Geometry Center at the University of Minnesota, USA. This Penrose tiling was generated by projecting a cross-section of a regular lattice in 5-dimensional space onto 3-dimensional space and by projecting the one-skeleton of a five-dimensional hypercube onto the 2-dimensional plane.
The area of fractal geometry has been around for many years but computer programs have made it possible to find fractals that lay hidden until recent years. Below are shown two images of one of the first computer generated fractals know as the Mandelbrodt Set named after a one of the pioneers of fractal imagery, Benoit Mandelbrodt.

VI. Dimensions and Hyperspace Motivated with Modern Art

A useful technique for introducing students to an abstract idea is to begin with examples in a more concrete familiar setting and then attempt to make progress to the more abstract idea. The works of art below illustrate various abstract ideas where each picture contains some unusual property. Since the paintings and sketches may not have been created with these particular mathematical ideas in mind, it may require some imagination on the part of the student to see the connection to mathematics.
| Figure 14. | Escher's Waterfall (Paradox) |
| Figure 15. | Piece of Bayeaux Tapestry – 11th Century (2 Dimensions - All Figures Appear Flat) |
| Figure 16. | Nude Descending Staircase – Marcel Duchamp (4th Dimension as Motion) |
| Figure 17. | Dora Maar – Pablo Picasso Person Viewed from 4 Dimensions |
| Figure 18. | Crucifixion – Salvador Dali 4 Dimensional Hypercube |
| Figure 19. | Persistence of Memory – Salvador Dali Melting Viewed as Homeomorphism |
VII. Non-Euclidean Geometry Motivated with Constructions and Escher Art

Most, if not all of the geometry that we use in daily life is based on the Euclidean geometry that is learned in high school. This experience gives us a solid geometric intuition for dealing with such notions as distance, size and angle. When the concept of non-Euclidean geometry is introduced to students, this new and revolutionary idea runs contrary to this lifetime of Euclidean intuition. In this situation, the use of non-Euclidean models with visual examples helps to make the transition easier.

The two works of art shown below are by the Dutch artist, M. C. Escher and are based on the non-Euclidean geometry of the Poincaré disk model of the hyperbolic plane. The two geometric constructions below Escher’s works show the Euclidean foundation for the paintings.

**Conclusion**

The motivation behind using visual imagery in the classroom is to draw on the natural graphical ability of the students of today and to utilize the modern computer techniques and applications at their disposal. The particular visual imagery examples given in this paper are designed to help students gain insight when learning abstract ideas. The expectation is that these visual techniques combined with a solid theoretical development of the mathematical material and a collaborative classroom setting will combine to create a productive learning environment.
References


