

USING HISTORY IN THE MATHEMATICS CLASSROOM: PYTHAGORAS, QUADRATICS, AND MORE

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Introduction

A number of years ago when I was a university student majoring in mathematics, I also had an interest in studying world history. One semester I was enrolling in a class in Russian history and noticed there were two classes offered with different professors. One of my classmates had taken courses from both of these teachers so I asked him to tell me about their individual teaching styles. My friend described to me how the first professor liked to view history from a space capsule high above the earth. In other words, this teacher was interested in a global perspective and investigated what was happening in different parts of the world during a given time period to understand the effect one part of the world had on another. The second professor was described as having a local perspective with my classmate likening his course to being in a forest with the class bumping into one tree named Catherine the Great, another named Ivan the Terrible, and another named Napoleon.

These two very different perspectives of teaching apply to all academic disciplines to some degree as well as to all grade levels of students. In mathematics courses at the high school and university level, there is a definite need to progress from one topic to another in order to cover a certain syllabus and for the students to gain certain skills and prepare for subsequent courses. This “bumping into trees”, to use my friend’s analogy, is useful to keep the class focused and to be able to assess whether or not the students are acquiring the desired learning outcomes.

One problem with this focused and local perspective of teaching mathematics is that students may not “see the forest through the trees”. The students may feel that mathematics is a never-ending sequence of topics, with different courses isolated from one other with only minor ideas in common. If on the other hand, students are allowed the opportunity to view mathematics from a global perspective, as a space flight high above the earth, then they may make connections between ideas and deepen their overall understanding. This global view of mathematics may also develop in students a deeper appreciation for the beauty of mathematics and to discover the rich variety, which is the history of mathematics.

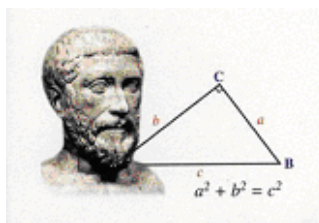


Figure 1 Pythagoras

Postcard 2001
World Mathematical Year

These stories from the history of mathematics are powerful tools for giving students this global perspective as is illustrated by the following example. Visual images in the form of postage stamps, photos, and other graphics are used to further develop the story. This example is a brief look at the historical development of two mathematical topics which are familiar to most high school and university students, namely, the Pythagorean Theorem (Figure 1) and the quadratic formula.

Many students are able to recite the quadratic formula from memory, “minus b plus or minus the square root of b squared minus four a times c all divided by two a ”. If you were to ask where this formula is useful, it might happen that many of the students who remembered it would not recall that it finds the roots of the quadratic equation $ax^2 + bx + c = 0$. Fewer still would have any idea where the quadratic formula originated.

If you ask about the Pythagorean Theorem, many students will remember the equation $x^2 + y^2 = z^2$ but may not recall its connection to geometry, which states, “for any right triangle, the square of the

hypotenuse equals the sum of the squares of the other two sides.” Also, the natural connection between the two ideas may not be evident.

So, what are the origins and history behind the Pythagorean Theorem and the quadratic formula? What roles have the various civilizations played in the development of these theorems? Perhaps more importantly for educators today, will introducing students to the history behind mathematical ideas give them a deeper understanding? The first two questions will be discussed here with the third question left for teachers to decide in their own individual classes.

The Babylonians

A cuneiform tablet, YBC 7289 (Figure 2), from the Yale Babylonian Collection, which dates from around 1800 B.C. to 1600 B.C. shows a square with numerals imprinted into it. The Babylonian number system had two main symbols, an arrowhead symbol pointing down represented one and an arrowhead pointing to the right represented ten. Their number system was sexagesimal, base 60, while each digit from 0 to 59 was written in base 10.



Figure 2 YBC 7289
 Yale Babylonian Collection
 Photo by Bill Casselman

The number written across the diagonal is $1 + \frac{24}{(60)} + \frac{51}{(60)^2} + \frac{10}{(60)^3} = 1.414212963$. This number is remarkably

close to $\sqrt{2} = 1.414213562\dots$, agreeing with it to five places beyond the decimal point. These numbers are so close that a strong argument can be made for the Babylonians to have known about the Pythagorean Theorem 1,000 years before Pythagoras was born.

The Egyptians



Figure 3 Ahmes Papyrus
 British Museum

The Berlin Papyrus (1800 B.C.) and Ahmes Papyrus (1650 B.C., Figure 3) contain several mathematics problems from ancient Egypt. These geometric problems involved lengths and area. In one problem you are given a square with an area of 100 square cubits and told that this area is equal to two smaller squares with the side of one of these squares is equal to $\frac{1}{2} + \frac{1}{4}$ of the other. The Egyptians reduced any fraction to a sum of fractions each of which has a one in the numerator.

This is equivalent to solving the system of two equations $x^2 + y^2 = 100$ and $y = \frac{3}{4}x$.

The solution is $x = 8$ and $y = 6$ which gives the Pythagorean triple $6^2 + 8^2 = 10^2$ and so the ancient Egyptians may have known of the Pythagorean Theorem as well.

Ancient China

Early mathematics in China developed in complete isolation from the rest of the world. In the ancient Chinese text, *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven*, (circa 600 B.C. or earlier) shows a diagram (Figure 4) of a proof of the Pythagorean Theorem based on calculating the area of a square in two different ways. If you label the sides of the right triangles x and y and the hypotenuse z , then

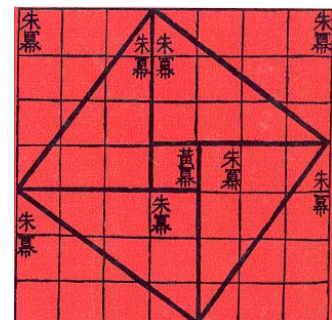


Figure 4 Chinese Proof
 Circular Paths of Heaven

the area of the large square is $(x + y)^2$. The area of the large square is also equal to the area of the

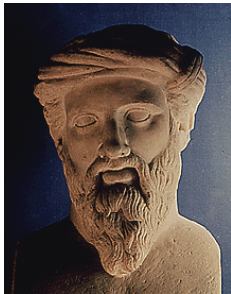


Figure 5
 Pythagoras
 Bettmann Archive

small square plus the areas of the four right triangles or $z^2 + 4\left(\frac{1}{2}xy\right)$.

Equating these, $(x + y)^2 = z^2 + 4\left(\frac{1}{2}xy\right)$ and simplifying leads to $x^2 + y^2 = z^2$ and the theorem is proved.

Ancient Greece

Pythagoras (c.580 – c.500 B.C., Figure 5) founded a school in Greece devoted

to the study of mathematics. His followers, the Pythagoreans, made the discovery that there were irrational numbers like the square root of two. This revelation was so contradictory to their geometric intuition that they kept it a secret. Pythagoras was involved with problems connected with the theorem

that bears his name but there is no solid evidence that he or members of his school had a rigorous proof of the theorem. In the 2,500 years since Pythagoras more that 350 different proofs of the theorem have been found.

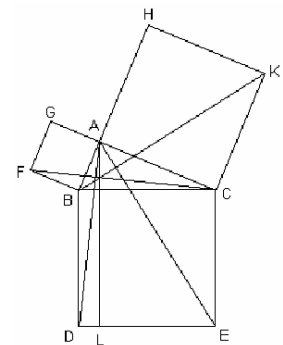


Figure 7
 "The Bride's Chair"

Euclid (circa 300 B.C.) wrote a geometry book, *Elements*, which was studied by students in its original form up until the 20th Century. In Book I, Proposition 47, of the *Elements*, Euclid gives a proof of the Pythagorean



Figure 6
 $3^2 + 4^2 = 5^2$
 Stamp
 Luxembourg 2000

Theorem. This proof contains a geometric diagram (Figure 6) often referred to as the "Bride's Chair". In Proposition 48, Euclid proves the converse of the Pythagorean Theorem which says that if a triangle with sides of lengths x , y , and z satisfy $x^2 + y^2 = z^2$, then the triangle must be a right triangle.

Diophantus of Alexandria (circa 300 A.D.) in his celebrated work, *Arithmetica*, developed a method for generating Pythagorean triples. These are sets of three positive integers, like 3, 4, 5 or 5, 12, 13 that satisfy $x^2 + y^2 = z^2$ (Figure 7). In

fact this problem by Diophantus to decompose a square given the sum of two other squares led the 17th French mathematician, Fermat, in the direction of his now famous Last Theorem. Diophantus was interested in integer solutions to many other problems, which are now referred to as Diophantine equations.

India

The great Indian mathematician, Brahmagupta (598 – c.665), took Diophantus' work a step further. Where Diophantus usually was concerned with finding one integer solution to an equation, Brahmagupta looked for methods of finding all solutions. Brahmagupta was interested in finding integer solutions to quadratic indeterminate equations of the form $ax^2 + c = y^2$, a

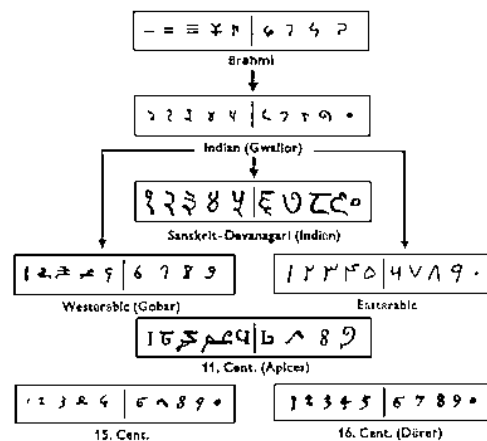


Figure 8 Hindu-Arabic Numerals
 Swami B. B. Visnu

variation of the Pythagorean equation. For the equation $92x^2 + 1 = y^2$, he found that the smallest integer solution is $x = 120$ and $y = 1151$, using some difficult arithmetic!



Figure 9 Al-Khwarizmi
 Stamp – Soviet Union 1983

One of Brahmagupta’s successors, Bhaskara (1114 – 1185), solved the equation $61x^2 + 1 = y^2$, to find that the smallest integer solution is $x = 226,153,980$ and $y = 1,766,319,049$, quite an accomplishment without a computer.

The Indian mathematicians of this time were instrumental in developing a number system, which included zero. This number system would make its way to the Islamic mathematicians in the Middle East and eventually evolve into the western number system of the present day (Figure 8) often referred to as Hindu-Arabic numerals.

The Islamic World

The 9th Century Islamic mathematician, Muhammad ibn Musa al-Khwarizmi (780 – 850), is considered the father of algebra (Figure 9). It is his landmark

work, *Hisab al-jabr w'al-muqabala*, from which the word algebra is derived. The origin of the quadratic formula lies with al-Khwarizmi’s early attempts to solve quadratic equations. A remarkable fact about al-Khwarizmi’s solutions is that he never used variables but rather solved everything in words. For example, a particular quadratic equation which al-Khwarizmi solved was $x^2 + 10x = 39$ (Figure 10). This, of course, was not the way that al-Khwarizmi viewed the equation but rather his description and solution of the equation went as follows.

$$5/2 + X + 5/2$$

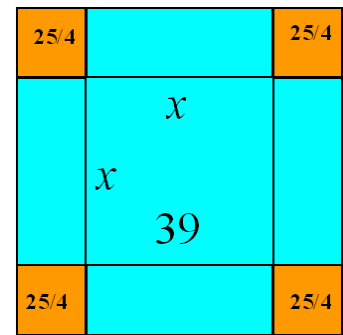


Figure 10

“Completing the Square”

... a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square, which combined with ten of its roots, will give a sum total of

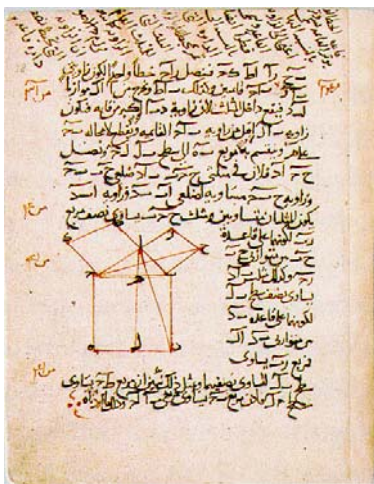


Figure 11 Arabic Proof
 Pythagorean Theorem
 13th Century Manuscript

39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this, which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square.

If al-Khwarizmi’s words are translated into a geometric picture, the technique of “completing square” comes to light. Further, if the geometry were translated into modern algebraic notation, the steps in the solution would look like $x^2 + 10x + 25 = 39 + 25 = 64$.

Then $(x+5)^2 = 64$ and $(x+5) = 8$ and so $x = 3$. If negative solutions are considered, al-Khwarizmi’s solution is just a short jump from the quadratic formula of today.

Besides developing a wealth of new mathematical ideas, Islamic mathematicians of the medieval period also were instrumental in

preserving many of the results previous civilizations. The 13th Century manuscript (Figure 11) by renowned Muslim mathematician, Nasir al-Din al-Tusi (1201 – 1274 A.D.), contains a proof of the Pythagorean Theorem. This proof complete with the “Bride’s Chair” diagram shows a variation of an ancient Greek proof by Euclid. This manuscript speaks loudly of the Golden Age of Mathematics in the Middle East when much of the mathematics of the ancient Greeks was preserved and expanded upon by brilliant mathematicians, astronomers and scientists from throughout the Islamic world.

Square Root Developments

Abraham bar-Hiyya Ha-Nasi (1060 – 1136 A.D.) was a Spanish mathematician and astronomer who wrote the earliest book in Europe that expanded on the algebra of the Islamic world. His work, *Treatise on Measurement and Calculation*, was the first text in Europe, which contained a complete solution of the general quadratic equation. It was translated into Latin as *Liber Embadorum* and published in 1145. In a coincidence of history, al-Khwarizmi’s algebra text was translated into Latin in this year as well and also included a complete solution of the quadratic.

A number of mathematicians worked on creating new notation for expressing solutions to the quadratic. In the 13th Century, Leonardo of Pisa (c. 1175 – 1250) used the Latin word “radix” for square root and this was shortened to the letter R with a slash through the right leg (Figure 12). In the 16th Century, the radical symbol appeared as a simple check mark (Figure 12) with expressions to be square rooted placed in parentheses. In his 1637 work, *La Geometrie*, René Descartes (1596 – 1650) added the horizontal bar to describe what expression was “under the radical”. From here our present day quadratic formula appears as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which are the solutions to the general quadratic $ax^2 + bx + c = 0$.



Figure 12
 Early Radical Signs

Extending Pythagoras

Around 1637, Pierre de Fermat (1601 – 1665), inspired by Diophantus’ search for positive integer solutions of $x^2 + y^2 = z^2$, began to search for similar solutions to $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$, and so



Figure 13

Pierre de Fermat and His Last Theorem
 Stamp – France 2001

on. Fermat had a habit of writing theorems and sketches of proofs in the margin of books he happened to be reading. In his copy of Diophantus’ *Arithmetica*, Fermat claimed that the equations $x^n + y^n = z^n$ (Figure 13) has no positive integer solutions for $n = 3, 4, 5 \dots$ and that he had a marvelous proof of this fact but the margin of the book was too small to contain it. Before he had a chance to elaborate on his proof, Fermat died.

In the centuries after Fermat, a number of mathematicians attempted to find a proof of this theorem. All of the other theorems, which Fermat stated in this fashion without proof, were indeed eventually proved and so this theorem about $x^n + y^n = z^n$ came to be known as Fermat’s Last Theorem. In 1770, Leonhard Euler (1707 – 1783) was able to prove that $x^3 + y^3 = z^3$ had no positive integer solutions. Fermat himself left a proof for $x^4 + y^4 = z^4$. In 1825, Adrien-Marie Legendre (1752 - 1833) jointly with Lejeune

Dirichlet (1805 - 1859) proved the case for $x^5 + y^5 = z^5$. Over time, solutions were found for increasing larger values for n . Later on, the theorem was proved for certain classes of prime exponents.

With the advent of computers, the equation $x^n + y^n = z^n$ was shown to have no positive integers solutions for very large values for n . With the increasing complexity of proofs for the different cases came the belief by many mathematicians that Fermat did not actually have a proof but only believed he did. In 1995, after six years of work in isolation, Princeton University professor Andrew Wiles (1953 –) announced that he had a proof of Fermat’s Last Theorem. After spending an additional year to modify his original proof, the proof has stood up to the scrutiny of the mathematics community and indeed sets the problem to rest. Fermat was correct when he wrote that $x^n + y^n = z^n$ has no now positive integer solutions (Figure 14) for $n = 3, 4, 5 \dots$.



Figure 14

Andrew Wiles Proves Fermat
 Stamp – Czech Republic 2000

Conclusion

This story of the development of the Pythagorean Theorem and the evolution of the quadratic formula is a story that spans 4,000 years with many civilizations and individual mathematicians playing pivotal roles. The history of mathematics is full of similar stories, in fact, for any mathematical topic that you can think of, there is most likely some sequence of discoveries combined with a cast of characters that carried it through to the present day. It remains for teachers to weave these rich stories into their classrooms to give students the complete picture of mathematics as a living, breathing, exciting, and adventurous endeavor of the human mind and spirit.

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