Problem. For $1 \le k \le n$, show that

$$(1) \qquad \sum_{i=0}^{k-1}\sum_{j=k}^{n}(-1)^{i+j-1}\binom{n}{i}\binom{n}{j}\frac{1}{j-i} = \sum_{i=0}^{k-1}\binom{n}{i}^2\left(H_{n-i}-H_i\right),$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the nth harmonic number and $H_0 = 0$.

Solution. We first prove the identities

$$(2)_n \qquad \sum_{j=k+1}^{n}(-1)^{k+j-1}\binom{n}{j}\frac{1}{j-k} = \binom{n}{k}(H_n - H_k) \quad \textit{for} \quad 0 \le k < n,$$

$$(3)_n \qquad \sum_{i=0}^{k-1}(-1)^{k+i-1}\binom{n}{i}\frac{1}{k-i} = \binom{n}{k}(H_n - H_{n-k}) \quad \textit{for} \quad 0 < k \le n$$

by induction on n. Let $L(n,k)$ denote the sum on the left-hand side of $(2)_n$. For $n = 1$, we have $L(1,0) = 1 = H_1$, as required. Now take $n > 1$. For $k = 0$, we use the recursion

$$(4) \qquad \binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$$

to break $L(n,0)$ into two sums, in the second of which we substitute

$$(5) \qquad \binom{n-1}{j-1}\frac{1}{j} = \binom{n}{j}\frac{1}{n},$$

obtaining

$$\begin{aligned} S(n,0) &= S(n-1,0) + \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j} \\ &= S(n-1,0) + \frac{1}{n}. \end{aligned}$$

Hence by induction $S(n,0) = H_n$, which proves $(2)_n$ in the case $k = 0$.

For $k > 0$, we again use (4) to break $S(n,k)$ into two sums, but now in the second one we set $j' = j - 1$ and write

$$S(n,k) = S(n-1,k) + \sum_{j'=k}^{n-1} (-1)^{k+j'} \binom{n-1}{j'} \frac{1}{j' - (k-1)}$$
$$= S(n-1,k) + S(n-1,k-1)\,.$$

Assuming inductively that $(2)_{n-1}$ holds, we have

$$S(n,k) = \binom{n-1}{k}\left(H_{n-1} - H_k\right) + \binom{n-1}{k-1}\left(H_{n-1} - H_{k-1}\right)$$
$$= \binom{n}{k}\left(H_{n-1} - H_k\right) + \binom{n-1}{k-1}\frac{1}{k}$$

and an application of (5) with j replaced by k yields $(2)_n$.

To derive $(3)_n$, we start with $(2)_n$, set $j = n - i$, and replace k by $n - k$, and $\binom{n}{n-m}$ by $\binom{n}{m}$, for $m = i$ and $m = k$. The result is $(3)_n$.

To prove (1), we use induction on k. Fix n and let $L(k)$ and $R(k)$ denote the left- and right-hand sides of (1), respectively. From $(2)_n$ with $k = 0$, we have $L(1) = H_n = R(1)$. Now observe that

$$L(k+1) - L(k) = \sum_{j=k+1}^{n} (-1)^{k+j-1} \binom{n}{k}\binom{n}{j} \frac{1}{j-k} - \sum_{i=0}^{k-1} (-1)^{i+k-1} \binom{n}{i}\binom{n}{k} \frac{1}{k-i}$$

for $1 \le k \le n-1$. Applying $(2)_n$ and $(3)_n$, we obtain

$$L(k+1) - L(k) = \binom{n}{k}^2 \left(H_{n-k} - H_k\right) = R(k+1) - R(k)\ ,$$

which by induction proves (1).

Jonathan Sondow
209 West 97 Street
New York, NY 10025
jsondow@alumni.princeton.edu