Problem. For $1 \leq k \leq n$, show that

$$\sum_{i=0}^{k-1} \sum_{j=k}^{n} (-1)^{i+j-1} \binom{n}{i} \binom{n}{j-i} \frac{1}{j-i} = \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the $n$th harmonic number and $H_0 = 0$.

Solution. We first prove the identities

$$\sum_{j=k+1}^{n} (-1)^{j-i-1} \binom{n}{j} \frac{1}{j-k} = \binom{n}{k} (H_n - H_k) \quad \text{for} \quad 0 \leq k < n,$$

$$\sum_{i=0}^{k-1} (-1)^{k+i-1} \binom{n}{i} \frac{1}{k-i} = \binom{n}{k} (H_n - H_{n-k}) \quad \text{for} \quad 0 < k \leq n$$

by induction on $n$. Let $L(n,k)$ denote the sum on the left-hand side of (2)$_n$. For $n = 1$, we have $L(1,0) = 1 = H_1$, as required. Now take $n > 1$. For $k = 0$, we use the recursion

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$$

to break $L(n,0)$ into two sums, in the second of which we substitute

$$\binom{n-1}{j-1} \frac{1}{j} = \binom{n}{j} \frac{1}{n},$$

obtaining

$$S(n,0) = S(n-1,0) + \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j}$$

$$= S(n-1,0) + \frac{1}{n}.$$

Hence by induction $S(n,0) = H_n$, which proves (2)$_n$ in the case $k = 0$. 
For \( k > 0 \), we again use (4) to break \( S(n,k) \) into two sums, but now in the second one we set \( j' = j - 1 \) and write

\[
S(n,k) = S(n-1,k) + \sum_{j'=k}^{n-1} (-1)^{k+j'} \binom{n-1}{j'} \frac{1}{j' - (k-1)}
\]

\[
= S(n-1,k) + S(n-1,k-1).
\]

Assuming inductively that \((2)_{n-1}\) holds, we have

\[
S(n,k) = \binom{n-1}{k}(H_{n-1} - H_k) + \binom{n-1}{k-1}(H_{n-1} - H_{k-1})
\]

\[
= \binom{n}{k}(H_{n-1} - H_k) + \binom{n-1}{k-1}\frac{1}{k}
\]

and an application of (5) with \( j \) replaced by \( k \) yields \((2)_n\).

To derive \((3)_n\), we start with \((2)_n\), set \( j = n - i \), and replace \( k \) by \( n - k \), and \( \binom{n}{m} \) by \( \binom{n}{m} \), for \( m = i \) and \( m = k \). The result is \((3)_n\).

To prove \((1)_n\), we use induction on \( k \). Fix \( n \) and let \( L(k) \) and \( R(k) \) denote the left- and right-hand sides of \((1)_n\), respectively. From \((2)_n\) with \( k = 0 \), we have \( L(1) = H_n = R(1) \). Now observe that

\[
L(k+1) - L(k) = \sum_{j=k+1}^{n} (-1)^{k+j-1} \binom{n}{k} \binom{n}{j-k} \frac{1}{j-k} - \sum_{i=0}^{k-1} (-1)^{j+k-1} \binom{n}{i} \binom{n}{k} \frac{1}{k-i}
\]

for \( 1 \leq k \leq n-1 \). Applying \((2)_n\) and \((3)_n\), we obtain

\[
L(k+1) - L(k) = \binom{n}{k}^2 (H_{n-k} - H_k) = R(k+1) - R(k),
\]

which by induction proves \((1)_n\).

Jonathan Sondow
209 West 97 Street
New York, NY 10025
jsondow@alumni.princeton.edu