CORRIGENDUM:
ON THE IRRATIONALITY OF SOME ALTERNATING SERIES

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The aim of this note is to point out that Theorem 1 of the first author's paper [**1**] is incorrect, and to replace it with Theorem A below and give an application.

Theorem 1. *Let* (a_n) *be a sequence of positive integers such that* $a_n(a_1a_2\cdots a_{n-1})^2 \to \infty$ *as* $n \to \infty$. *Then the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n(a_1\cdots a_{n-1})^2}$ *is irrational.*

The constant sequence $(a_n) = 2, 2, \ldots$ is a counterexample. The mistake in the proof lies in assuming that, with $u_i = (a_1\cdots a_{i-1})^{-2}$ and $v_i = a_i$, the sum $\sum_{i=1}^{n}(-1)^{i-1}u_i/v_i$ is a rational number with denominator $v_1\cdots v_n$. In fact, the denominator is $v_n(v_1\cdots v_{n-1})^2$.

The following result is a generalization of Lemma 1 in [**1**].

Theorem A. *Let* $(r_n) = (h_n/k_n)$ *be a sequence of rational numbers, with* $k_n > 0$, *satisfying* (*i*) $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$ *and* (*ii*) $\liminf_{n\to\infty} k_n|r_{n+1} - r_n| = 0$. *Then the alternating series*

$$r_1 - (r_1 - r_2) + (r_3 - r_2) - (r_3 - r_4) + \cdots$$

converges and its sum is irrational.

Proof. It follows from (*i*) and (*ii*) that the conditions of Leibniz's alternating series test are satisfied. Thus the series converges and its sum, θ, lies between the partial sums r_n and r_{n+1}, for $n = 1, 2, \ldots$. Suppose now that $\theta = a/b$ is rational, $b > 0$. Then (*ii*) and the inequalities $0 < |\theta - r_n| < |r_{n+1} - r_n|$ imply that $0 < |ak_n - bh_n| < bk_n|r_{n+1} - r_n| < 1$, for some $n \geq 1$. This contradicts the fact that $ak_n - bh_n$ is an integer, completing the proof. ■

As an application of Theorem A (or of Lemma 1), we obtain a new proof that *if* p_n/q_n *is the n-th convergent of an infinite simple continued fraction,* $n = 0, 1, 2, \ldots$, *then the sum of the series* $p_0/q_0 + \sum_{n=0}^{\infty}(-1)^n/(q_n q_{n+1})$ *is an irrational number,* namely, the value of the continued fraction.

REFERENCE

1. J. Sándor, *On the irrationality of some alternating series*, Studia Univ. Babes-Bolyai **33** (1988) 8-12.

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