Problem. Fix an integer $B \geq 2$ and let $s_B(n)$ denote the sum of the digits of an integer $n$ in base $B$. Prove the infinite product formula

$$\prod_{n=0}^{\infty} \prod_{0 < k < B} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{s_B(n)}} = \frac{1}{\sqrt{B}}.$$  

Solution. Set

$$\varepsilon(n) = \varepsilon_B(n) = (-1)^{s_B(n)}.$$  

Notice that $\varepsilon(2n + 1) = -\varepsilon(2n)$ for all $n \geq 0$. (Proof. If $B$ is odd, clearly $(-1)^{s_B(n)} = (-1)^n$. If $B$ even, and if $d_1$ is the 1’s digit of $2n$ in base $B$, then $d_1 < B - 1$, so $s_B(2n + 1) = s_B(2n) + 1$. Hence in both cases $\varepsilon(2n + 1) = -\varepsilon(2n)$.) It follows that the infinite product converges, because we can write it as a product over $n \geq 0$ of factors of the form

$$\left( \frac{B(2n) + k + 1}{B(2n) + k} \cdot \frac{B(2n + 1) + k}{B(2n + 1) + k + 1} \right)^{\pm 1} = \left( 1 + \frac{B}{(B(2n) + k)(B(2n + 1) + k + 1)} \right)^{\pm 1}.$$  

Note that if $0 \leq k < B$, then $s_B(Bn + k) = s_B(n) + k$, so that

$$\varepsilon(Bn + k) = (-1)^k \varepsilon(n).$$  

Now let

$$\delta_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$
and observe that the following product of products telescopes:

\[
\prod_{k=0}^{B-1} \prod_{n=\delta_k}^{\infty} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\epsilon(n)} = \left( \frac{1}{2} \frac{2}{3} \cdots \frac{B-1}{B} \right)^{\epsilon(0)} \prod_{n=1}^{\infty} \left( \frac{Bn}{Bn + 1} \frac{Bn + 1}{Bn + 2} \cdots \frac{Bn + B - 1}{Bn + B} \right)^{\epsilon(n)}
\]

\[
= \frac{1}{B} \prod_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^{\epsilon(n)}
\]

(1)

If we split the last product, collecting factors with the same index \( n \) modulo \( B \), we obtain another product of products:

\[
\prod_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^{\epsilon(n)} = \prod_{k=0}^{B-1} \prod_{n=\delta_k}^{\infty} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\epsilon(Bn+k)} = \prod_{k=0}^{B-1} \prod_{n=\delta_k}^{\infty} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^k \epsilon(n)}
\]

Substituting this into (1), and noting that the infinite products are all nonzero (by virtue of being convergent and having no zero factors), we see that those with even \( k \) cancel, leaving

\[
\prod_{k \text{ odd}} \prod_{n=0}^{\infty} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\epsilon(n)} = \frac{1}{B} \prod_{k \text{ odd}} \prod_{n=0}^{\infty} \left( \frac{Bn + k}{Bn + k + 1} \right)^{-\epsilon(n)}
\]

All the products are positive, and the desired formula follows.

Comment. The case \( B = 2 \) is due to D. R. Woods and D. Robbins; see