Problem Submission

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Problem. Fix an integer $B \geq 2$ and let $s_B(n)$ denote the sum of the digits of an integer n in base B. Prove the infinite product formula

$$\prod_{n=0}^{\infty} \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^{s_B(n)}} = \frac{1}{\sqrt{B}}.$$

Solution. Set

$$\varepsilon(n) = \varepsilon_B(n) = (-1)^{s_B(n)}.$$

Notice that $\varepsilon(2n+1) = -\varepsilon(2n)$ for all $n \geq 0$. (*Proof*. If B is odd, clearly $(-1)^{s_B(n)} = (-1)^n$. If B even, and if d_1 is the 1's digit of $2n$ in base B, then $d_1 < B-1$, so $s_B(2n+1) = s_B(2n)+1$. Hence in both cases $\varepsilon(2n+1) = -\varepsilon(2n)$.) It follows that the infinite product converges, because we can write it as a product over $n \geq 0$ of factors of the form

$$\left(\frac{B(2n)+k+1}{B(2n)+k} \cdot \frac{B(2n+1)+k}{B(2n+1)+k+1} \right)^{\pm 1} = \left(1 + \frac{B}{\big(B(2n)+k\big)\big(B(2n+1)+k+1\big)} \right)^{\pm 1}.$$

Note that if $0 \leq k < B$, then $s_B(Bn+k) = s_B(n)+k$, so that

$$\varepsilon(Bn+k) = (-1)^k \varepsilon(n).$$

Now let

$$\delta_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and observe that the following product of products telescopes:

$$\prod_{k=0}^{B-1}\prod_{n=\delta_k}^{\infty}\left(\frac{Bn+k}{Bn+k+1}\right)^{\varepsilon(n)} = \left(\frac{1}{2}\frac{2}{3}\cdots\frac{B-1}{B}\right)^{\varepsilon(0)}\prod_{n=1}^{\infty}\left(\frac{Bn}{Bn+1}\frac{Bn+1}{Bn+2}\cdots\frac{Bn+B-1}{Bn+B}\right)^{\varepsilon(n)}$$

$$= \frac{1}{B}\prod_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{\varepsilon(n)}. \tag{1}$$

If we split the last product, collecting factors with the same index n modulo B, we obtain another product of products:

$$\prod_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{\varepsilon(n)} = \prod_{k=0}^{B-1}\prod_{n=\delta_k}^{\infty}\left(\frac{Bn+k}{Bn+k+1}\right)^{\varepsilon(Bn+k)} = \prod_{k=0}^{B-1}\prod_{n=\delta_k}^{\infty}\left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^k\,\varepsilon(n)}.$$

Substituting this into (1), and noting that the infinite products are all nonzero (by virtue of being convergent and having no zero factors), we see that those with even k cancel, leaving

$$\prod_{\substack{k \text{ odd}\\ 0<k<B}}\prod_{n=0}^{\infty}\left(\frac{Bn+k}{Bn+k+1}\right)^{\varepsilon(n)} = \frac{1}{B}\prod_{\substack{k \text{ odd}\\ 0<k<B}}\prod_{n=0}^{\infty}\left(\frac{Bn+k}{Bn+k+1}\right)^{-\varepsilon(n)}.$$

All the products are positive, and the desired formula follows.

Comment. The case $B = 2$ is due to D. R. Woods and D. Robbins; see

D. R. Woods, Problem proposal E2692, *Amer. Math. Monthly* **85** (1978) 48;
D. Robbins, Solution to problem E2692, *Amer. Math. Monthly* **86** (1979) 394-395.