Let $\alpha_0(s) = \sup \{a: N^{s-1} \sum_{n=1}^{N} n^{-s} = \int_0^1 x^{-s} dx + O(N^{-a})\}$, and let $\alpha_1(s) = \sup \{a: N^{s-1} \sum_{n=1}^{N} n^{-s} \log(n/N) = \int_0^1 x^{-s} \log x dx + O(N^{-a})\}$. By using the Euler-Maclaurin formula [cf. H. M. Edwards, Riemann’s zeta function, Academic Press, New York-London, 1974; MR0466039 (57 #5922)], the author deduces that, for $s \neq 0, 1$, $\alpha_0(s)$ is equal to 1 when $\zeta(s) = 0$, and is equal to $\min(1, 1 - \Re s)$ otherwise. It follows that the Riemann hypothesis is equivalent to $\alpha_0(s)$ being continuous on either the strip $0 < \Re s < \frac{1}{2}$ or $\frac{1}{2} < \Re s < 1$. The author also shows that $\alpha_1(s)$ is continuous on the open critical strip if and only if $\zeta(s)$ has only simple zeros.

Reviewed by Xian-Jin Li

© Copyright American Mathematical Society 1998, 2008