MR2286247 (Review) 11Y60 (11J72 33D15)
Sondow, Jonathan; **Zudilin, Wadim [Zudilin, V. V.]** (RS-MOSC-MM)
Euler's constant, q-logarithms, and formulas of Ramanujan and Gosper. (English summary)
Ramanujan J. **12** (2006), *no. 2,* 225–244.

In [Proc. Amer. Math. Soc. **131** (2003), no. 11, 3335–3344 (electronic); MR1990621 (2004b:11102)] the first author gave the following construction of $\mathbb{Z}$-linear forms involving Euler's constant $\gamma=\lim_{n\to\infty}(\sum_{k=1}^n 1/k-\log n)$ and logarithms:

$$(*)\quad d_{2n}I_n\in\mathbb{Z}+\mathbb{Z}\gamma+\mathbb{Z}\log(n+1)+\mathbb{Z}\log(n+2)+\cdots+\mathbb{Z}\log(2n),$$

where

$$I_n=\iint_{[0,1]^2}\frac{x^n(1-x)^ny^n(1-y)^n}{(1-xy)|\log xy|}\,dx\,dy,$$

d_n denotes the least common multiple of the numbers $1,2,\dots,n$, and $\lim_{n\to\infty}d_{2n}I_n=0$. This type of approximation allows one to deduce some irrationality criteria for Euler's constant γ, for which irrationality has not yet been proved.

In the paper under review, using a q-analog of the logarithm

$$\ln_q(1+z)=\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{q^\nu-1},\qquad |z|<|q|,$$

with $q>1$, the authors prove an inclusion analogous to $(*)$:

$$d_{2^n}I_{n,2^n}\in\mathbb{Z}+\mathbb{Z}\gamma+\mathbb{Z}\log 2+\mathbb{Z}\ln_2\left(1+\frac{1}{2^n}\right)$$
$$+\mathbb{Z}\ln_2\left(1+\frac{2}{2^n}\right)+\cdots+\mathbb{Z}\ln_2\left(1+\frac{2^n-1}{2^n}\right),$$

where

$$I_{n,m}=\int_0^1\frac{(1-x)^m}{(1+x)}\sum_{\nu=n+1}^{\infty}x^{2^\nu-1}\,dx$$

and $\lim_{n\to\infty}d_{2^n}I_{n,2^n}=0$. Using this inclusion, the authors give irrationality criteria for Euler's constant as well as asymptotic formulas for computing γ. (Note that another extension of Sondow's approach can be found in the reviewer's papers [K. Hessami Pilehrood and T. Hessami Pilehrood, Math. Z. **255** (2007), no. 1, 117–131; MR2262724; J. Number Theory **108** (2004), no. 1, 169–185; MR2078662 (2005e:11095)].)

On the other hand, the authors prove some new formulas for Euler's constant. It should be noted that the quantity $I_{n,m}$ is a generalization of the famous Catalan integral $I_{0,0}$ for γ. Using this integral the authors give a new proof of Gosper's acceleration of Vacca's "base 2" series for Euler's constant and then generalize Gosper's series to base q. They also attempt to find a generalization of integral $I_{n,m}$ to base q and succeed only for the case $q = 3$.

Reviewed by *Khodabakhsh Hessami Pilehrood*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.