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A geometric proof that e is irrational and a new measure of its irrationality.
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From the introduction: "While there exist geometric proofs of irrationality for $\sqrt{2}$ [T. M. Apostol, Amer. Math. Monthly **107** (2000), no. 9, 841–842; Zbl 0990.11046; B. Turner, Math. Mag. **50** (1977), no. 5, 263], no such proof for e, π, or $\ln 2$ seems to be known. In section 2 we use a geometric construction to prove that e is irrational. The proof leads in section 3 to a new measure of irrationality for e, that is, a lower bound on the distance from e to a given rational number, as a function of its denominator. A connection with the greatest prime factor of a number is discussed in section 4. In section 5 we compare the new irrationality measure for e with a known one and state a number-theoretic conjecture that implies that the known measure is almost always stronger. The new measure is applied in section 6 to prove a special case of a result from ["Which partial sums of the Taylor series for e are convergents to e? (And a link to the primes $2, 5, 13, 37, 463, \ldots$.) With an appendix by Kyle Schalm", preprint, arxiv.org/abs/0709.0671], leading to another conjecture. Finally, in section 7 we recall a theorem of G. Cantor that can be proved by a similar construction."

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.