MR2319662 (2008f:40013) 40A30 (11Y60 33B15 33B30)

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The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant. (English summary)

J. Math. Anal. Appl. **332** (2007), *no. 1,* 292–314.

Summary: "We define the generalized-Euler-constant function $\gamma(z) = \sum_{n=1}^{\infty} z^{n-1}(\frac{1}{n} - \log\frac{n+1}{n})$ when $|z| \le 1$. Its values include both Euler's constant $\gamma = \gamma(1)$ and the 'alternating Euler constant' $\log\frac{4}{\pi} = \gamma(-1)$. We extend Euler's two zeta-function series for γ to polylogarithm series for $\gamma(z)$. Integrals for $\gamma(z)$ provide its analytic continuation to $\mathbb{C} - [1, \infty]$. We prove several other formulas for $\gamma(z)$, including two functional equations; one is an inversion relation between $\gamma(z)$ and $\gamma(1/z)$. We generalize Somos's quadratic recurrence constant and sequence to cubic and other degrees, give asymptotic estimates, and show relations to $\gamma(z)$ and to an infinite nested radical due to Ramanujan. We calculate $\gamma(z)$ and $\gamma'(z)$ at roots of unity; in particular, $\gamma'(-1)$ involves the Glaisher-Kinkelin constant A. Several related series, infinite products, and double integrals are evaluated. The methods used involve the Kinkelin-Bendersky hyperfactorial K function, the Weierstrass products for the gamma and Barnes G functions, and Jonquière's relation for the polylogarithm."

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.