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Criteria for irrationality of Euler's constant. (English summary)
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In this interesting paper, the author considers a sequence of integrals similar to the famous integrals related to $\zeta(2)$ and $\zeta(3)$ introduced by Beukers, namely:

$$I_n = -\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)\log(xy)}dxdy.$$

He remarks that, for any integer $n \geq 1$, $I_n = \binom{2n}{n}\gamma + L_n + A_n$, where γ is Euler's constant, $d_{2n}L_n \in \mathbb{Z}\log(n) + \mathbb{Z}\log(n+1) + \cdots + \mathbb{Z}\log(2n)$, $d_{2n}A_n$ is an integer and d_{2n} =l.c.m.$(1, 2, \ldots, 2n)$. Amongst other things, he shows that if the fractional part of $d_{2n}L_n$ satisfies

(i) $$\{d_{2n}L_n\} \geq 2^{-n} \quad \text{for infinitely many } n$$

(a condition for which he also gives numerical evidence), then γ is irrational. Condition (i) suggests that Baker's theory of linear forms in logarithms of algebraic numbers could be useful. Unfortunately, using the pigeonhole principle, the author has recently constructed a sequence of linear forms in logarithms, similar to L_n, which fails to satisfy (i). Hence, if a proof of (i) exists it must use particular properties of L_n and not be based solely on the general theory of these forms.

Reviewed by *Tanguy Rivoal*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.