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Allouche, J.-P.; **Shallit, J.** (3-WTRL-SC); **Sondow, J.**
Summation of series defined by counting blocks of digits. (English summary)
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In a recent paper, Sondow ["New Vacca-type rational series for Euler's constant and its 'alternating' analogue $\ln\frac{4}{\pi}$", preprint, arxiv.org/abs/math.NT/0508042] gave the following two formulas:

$$\gamma^{\pm}=\sum_{n\geq 1}\frac{N_1(n)\pm N_0(n)}{2n(2n+1)},$$

where $\gamma^+=\gamma=0.57721\ldots$ is the Euler constant, $\gamma^-=\log\frac{4}{\pi}=0.24156\ldots$ is the "alternating Euler constant", and $N_1(n)$ (respectively $N_0(n)$) is the number of 1's (respectively 0's) in the binary expansion of the integer n. In particular, it follows that

$$\sum_{n\geq 1}\frac{N_1(n)}{2n(2n+1)}=\frac{\gamma+\log\frac{4}{\pi}}{2}.$$

The purpose of the paper under review is to show that the above result can be deduced from a general lemma given by Allouche and Shallit [in *Analytic number theory (Tokyo, 1988)*, 19–30, Lecture Notes in Math., 1434, Springer, Berlin, 1990; MR1071742 (91i:11111)]. Furthermore, the two series

$$\sum_{n\geq 1}\frac{N_{w,2}(n)}{2n(2n+1)}\quad\text{and}\quad\sum_{n\geq 1}\frac{N_{w,2}(n)}{2n(2n+1)(2n+2)}$$

are summed, where $N_{w,B}(n)$ counts the number of occurrences of the block w in the B-ary expansion of the integer n. Finally, a generalization of these two results is given, including an extension to base $B>2$, as well as a method for giving alternate proofs without using the general lemma referred to above.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.