Epimorphisms and dominions in varieties of lattices

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Summary

Let $\mathbf{V}$ be a variety of lattices, and let $L \subseteq M$ be lattices in $\mathbf{V}$. The *dominion* of L in M is defined as the sublattice consisting of all $x \in M$ such that for all homomorphisms $f : M \to M'$ with $M' \in \mathbf{V}$, $f(x)$ is determined by $f|_L$. It is denoted $\mathrm{dom}_M^{\mathbf{V}}(L)$. We say that the dominion of L in M is *trivial* if $\mathrm{dom}_M^{\mathbf{V}}(L) = L$. The variety of distributive lattices has nontrivial dominions, which are charcterized below, but in any larger variety of lattices, all sublattices of distributive lattices have trivial dominion. There are uncountably many varieties with nonsurjective epimorphisms, including every variety with a finite bound on the length (or on the width) of its subdirectly irreducible members, and every almost distributive variety. We will find many varieties in which any epimorphically embedded sublattice must be cofinal. For any finite partition lattice M, there is a distributive sublattice D such that $\mathrm{dom}_M^{\mathrm{Var}(M)}(D) = M$. We find an example a two-element sublattice with an infinite dominion.

Introduction

We use the notation $\mathrm{dom}_M^{\mathcal{C}}(L)$ for the dominion of the subalgebra $L \subseteq M$ with respect to the category $\mathcal{C}$. Every category considered here will be assumed to be a full subcategory of a variety of algebras. Recall the definition: $\mathrm{dom}_M^{\mathcal{C}}(L) =$

$$\{x \in M \,|\, \forall A \in \mathcal{C}, \forall \text{ morphisms } f, g : M \to A, f|_L = g|_L \Rightarrow f(x) = g(x)\}.$$

Note this defines a closure operator on the subalgebras of M. The dominion of L will be called *trivial* if it is equal to L; in this case we also say that L is *dominion-closed* in M. We say L is *dense* if its dominion is M, i.e. if the inclusion of L in M is an epimorphism.

Some important trivial lemmas

Lemma 1 *([5]) The dominion is functorial, in the sense that if $A \subseteq B$, $A' \subseteq$*

B', and $f : B \to B'$ is a homomorphism such that $f(A) \subseteq A'$, then f maps the dominion of A into the dominion of A'.

The proof is trivial, as is the proof of this related lemma:

Lemma 2 *Let $f : A \to A'$ be a homomorphism in $\mathcal{C}$.*

(i) For any dominion-closed subalgebra $B' \subseteq A'$, $f^{-1}(B')$ is dominion-closed in A.

(ii) For any subalgebra $B \subseteq A$, $\mathrm{dom}_A^{\mathcal{C}}(B) \subseteq f^{-1}(\mathrm{dom}_{A'}^{\mathcal{C}}(f(B)))$.

Suppose $\mathcal{C}$ has coproducts and is closed under homomorphic images. Then Lemma 1.1 of [5] shows that if $d \in \mathrm{dom}_A^{\mathcal{C}}(B)$, then there is a "computational proof" of this fact, which involves only finitely many elements of A. This proves

Lemma 3 *If $\mathcal{C}$ has coproducts, and is closed under subalgebras and homomorphic images, and $d \in \mathrm{dom}_B^{\mathcal{C}}(A)$, then there is a finitely generated subalgebra $F \subseteq B$, and a finitely generated subalgebra $F' \subseteq F \cap A$ such that $d \in \mathrm{dom}_F^{\mathcal{C}}(F')$.*

Lemma 4 *(cf. Exercise 3.12 of [1]) Suppose $\mathcal{C} = \mathbf{ISP}(K)$ for some class of algebras K. Then $\mathrm{dom}_M^{\mathcal{C}}(L) =$*

$$\{x \in M \mid \forall A \in K, \forall \text{ morphisms } f, g : M \to A, f|_L = g|_L \Rightarrow f(x) = g(x)\}.$$

Proof: It is clear that the left-hand side is contained in the right-hand side. Suppose $x \in M$ is not in $\mathrm{dom}_M^{\mathcal{C}}(L)$. Then there is $A \in \mathcal{C}$ and $f, g : M \to A$ such that $f|_L = g|_L$ but $f(x) \neq g(x)$. Since A can be embedded in a product of algebras in K, there is $A' \in K$ and $h : A \to A'$ such that h separates $f(x)$ and $g(x)$. Then $hf, hg : M \to A'$ are homomorphisms that agree on L but not on x. Therefore x is not in the right-hand side. $\square$

For example, if $\mathcal{C}$ is a variety $\mathbf{V}$, we can choose $K = \mathbf{V}_{\mathrm{SI}}$ (or some subset, if some subdirectly irreducible algebras embed in others.) This shows that in a finitely generated congruence distributive variety, the dominion of a subalgebra of a finite algebra can be determined by a finite computation.

Distributive lattices

We say that a sublattice $L \subseteq D$ is *closed under relative complementation* if whenever there are elements $a < b < c$ in L, and $x \in D$ such that x is the relative complement of b in $[a, c]$, then $x \in L$.

In [3] it was determined exactly when strong amalgamation can occur in the variety of distributive lattices (which we write as $\mathbf{D}$). From this result, the characterization of dominions in $\mathbf{D}$ follows trivially:

Theorem 5 *$\mathrm{dom}_D^{\mathbf{D}}(L) = L$ if and only if L is closed under relative complementation. Thus $\mathrm{dom}_D^{\mathbf{D}}(L)$ is the smallest sublattice of D that contains L and is closed under relative complementation.*

In contrast with this result, we will prove that in any variety of lattices larger than $\mathbf{D}$, sublattices of distributive lattices have trivial dominion. First we prove a special case:

Lemma 6 *Let*

$$L = \{(0,0),(1,0),(1,1)\} \subseteq \mathbf{2}\times\mathbf{2}$$

and let $\mathbf{V}$ *be any variety larger than* $\mathbf{D}$*. Then*

$$\mathrm{dom}^{\mathbf{V}}_{\mathbf{2}\times\mathbf{2}}(L) = L.$$

Proof: Since $\mathbf{V}$ contains a nondistributive lattice, and is closed under subalgebras, $\mathbf{V}$ contains either M_3 or N_5. It is easy to see that there are two homomorphisms from $\mathbf{2}\times\mathbf{2}$ into either M_3 or N_5 that agree on L but not on $(0,1)$. □

Lemma 7 *(Theorem 1 of [6]) Let* L *be a distributive lattice with* 0 *and* 1*, and let* L' *be a sublattice that contains* 0 *and* 1*. For any* $a \in L\setminus L'$*, there exist prime ideals* I_1 *and* I_2 *of* L *such that* $I_1 \cap L' \subseteq I_2 \cap L'$*,* $a \in I_1$*, and* $a \notin I_2$*.*

For any lattice L let L_{01} denote the lattice formed by adding a new 0 and 1 to L.

Corollary 8 *Let* $L' \subseteq L$ *be distributive lattices. Then there is a set* S *of homomorphisms* $L \to \mathbf{2}\times\mathbf{2}$ *such that*

$$L' = \bigcap_{f\in S} f^{-1}(\{(0,0),(1,0),(1,1)\}).$$

Proof: We apply Lemma 7 to ${L'}_{01} \subseteq L_{01}$. For any $a \in L\setminus L'$ (or equivalently, in $L_{01} \setminus {L'}_{01}$), let I_1 and I_2 be the ideals given by the lemma. Define $f_a : L \to \mathbf{2}\times\mathbf{2}$ so that $f(x) = (x_1, x_2)$, where x_i is 0 if $x \in I_i$, and 1 if $x \notin I_i$. Then f_a is a homomorphism, and $L' \subseteq {f_a}^{-1}(\{(0,0),(1,0),(1,1)\})$, but $a \notin {f_a}^{-1}(\{(0,0),(1,0),(1,1)\})$. Therefore

$$S = \{f_a \,|\, a \in L \setminus L'\}$$

is the required set. □

Theorem 9 *Let* L' *be a sublattice of a distributive lattice* L*. Let* $\mathbf{V}$ *be a variety larger than* $\mathbf{D}$*. Then* $\mathrm{dom}^{\mathbf{V}}_{L}(L') = L'$*.*

Proof: By Corollary 8, L' is the intersection of inverse images of

$$\{(0,0),(1,0),(1,1)\} \subseteq \mathbf{2}\times\mathbf{2}.$$

By Lemma 6, $\{(0,0),(1,0),(1,1)\}$ is dominion-closed; so by Lemma 2, the inverse images are dominion-closed. Therefore L is the intersection of dominion-closed sublattices, and thus is dominion-closed. □

Given a nondistributive lattice $L \in \mathbf{V}$, let $f : L \to D$ be the universal map from L into a distributive lattice. Using Theorem 9 and Lemma 2(ii), we get that $\mathrm{dom}_L^{\mathbf{V}}(L') \subseteq f^{-1}(f(L'))$ for any $L' \subseteq L$. In computing the dominions of sublattices of L, we can ignore pairs of maps that both have distributive image, because if such a pair agrees on L', it must agree on $f^{-1}(f(L'))$.

Almost distributive lattices

We will use the symbol $\mathbf{A}$ for the variety of almost distributive lattices, which is defined by four identities that can be found in [7]. The next two lemmas give the properties of $\mathbf{A}$ that we need.

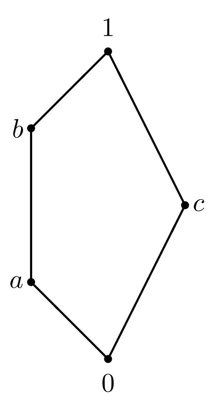
Lemma 10 *(Lemma 4.11 of [7]) For any distributive lattice D and $d \in D$, $D[d]$ is almost distributive.*

Here $D[d]$ is the lattice obtained from D by doubling d. Let γ denote the natural map from $D[d]$ onto D.

Lemma 11 *(Corollary 4.14 of [7]) A subdirectly irreducible lattice L is almost distributive if and only if $L \cong D[d]$ for some distributive lattice D and $d \in D$.*

Theorem 12 *There is an inclusion in $\mathrm{Var}(N_5)$ that is a nonsurjective epimorphism in $\mathbf{A}$.*

Proof: We label the elements of N_5 in the diagram.



Let

$$L = (N_5 \times N_5) \setminus (\{0, c\} \times \{c, 1\})$$

and let $S = (N_5 \times N_5) \setminus (\{0, c\} \times \{b, c, 1\})$. A simple computation shows that L and S are both sublattices of $N_5 \times N_5$. We will show that the inclusion of S in L is an epimorphism.

Let $D[d]$ be any subdirectly irreducible almost distributive lattice, and let g be any homomorphism from L to $D[d]$. Let p_1 and p_2 be the restrictions to L of the two coordinate projections $N_5 \times N_5 \to N_5$.

Claim: If g does not collapse $[(1,a),(1,b)]$, then it is the composite of p_2 with some injection $N_5 \hookrightarrow D[d]$.

Proof of claim: If g does not collapse $[(1,a),(1,b)]$, then it also does not collapse $[(a,a),(a,b)]$ or $[(b,a),(b,b)]$. The sublattices $\{a\} \times N_5$, $\{b\} \times N_5$, and $\{1\} \times N_5$ are subdirectly irreducible, and since their critical quotients are not collapsed, these quotients are all mapped to $[(d,0),(d,1)]$. Then

$$g((c,b)) \vee (d,0) = g((c,b)) \vee g((b,a)) = g((1,b)) = (d,1)$$

but $(d,1)$ is join irreducible, so $g((c,b)) = (d,1)$. Then

$$g((0,b)) = g((c,b)) \wedge g((a,b)) = (d,1).$$

So g collapses $[(0,b),(1,b)]$; by perspectivity it also collpases $[(0,a),(1,a)]$ and $[(0,0),(1,0)]$. Since g collapses $[(a,b),(1,b)]$, it collapses $[(a,1),(1,1)]$, and thus also $[(a,c),(1,c)]$. This proves the claim.

We call the maps described in the claim *type 2* maps. Now note that if we switch the components of $N_5 \times N_5$ and then dualize it, L is preserved. So by this symmetry, we see that if g does not collapse $[(a,0),(b,0)]$, then it is the composite of p_1 with some injection $N_5 \hookrightarrow D[d]$. Call such maps *type 1.* Then any map that is neither type 1 nor type 2 must collapse both $[(a,0),(b,0)]$ and $[(1,a),(1,b)]$, and therefore it has distributive image. Call these *type 3.*

Now let $g, h : L \to D[d]$ be two maps that agree on S. If both have distributive image, then since they agree on $(0,a)$ and (c,a), they also agree on $(0,b)$ and (c,b), and thus are equal. If g is type 1 or type 2, then $g|_S$ has nondistributive image, so g cannot agree on S with a map of type 3. If g is type 1 and h is type 2, they do not agree on (a,b). If both are type 1, or both are type 2, then clearly they are equal. □

Corollary 13 *Every subvariety of* **A** *has nonsurjective epimorphisms.*

Proof: We have already found nonsurjective epimorphisms in **D**. Since M_3 is not almost distributive, any larger subvariety of **A** must contain $\mathrm{Var}(N_5)$. □

Many more varieties with nonsurjective epimorphisms

Lemma 14 *Let* **V** *be a prevariety generated by lattices of length less than* $2n$, *and suppose* $S \in \mathbf{V}$ *is a subdirectly irreducible lattice of length* n. *Let*

$$L = \{(x,y) \in S \times S \mid x \leq y\}$$

let $[a, b]$ be any critical quotient of S, and let L' be the sublattice of $S \times S$ generated by $L \cup \{(b, a)\}$. Then the inclusion of L in L' is an epimorphism in $\mathbf{V}$.

Proof: Let M be a lattice of length less than $2n$, and let $f : L' \to M$ be a homomorphism. Since L' has greater length than M, f is not injective. In particular, the sublattices $\{0\} \times S$ and $S \times \{1\}$, which are both subdirectly irreducible, cannot both be embedded in M by f. So f collapses either $[(0, a), (0, b)]$ or $[(a, 1), (b, 1)]$, and therefore it collapses either $[(a, a), (a, b)]$ or $[(a, b), (b, b)]$. Therefore $f((b, a))$ is determined by $f((a, a))$, $f((a, b))$, and $f((b, b))$. □

Theorem 15 *For any $n > 1$, let $\mathbf{L}^n_\infty$ be the variety generated by all lattices of length less than or equal to n. Then any subvariety of $\mathbf{L}^n_\infty$ has nonsurjective epimorphisms.*

Proof: Let $\mathbf{V}$ be a subvariety of $\mathbf{L}^n_\infty$; we may assume that $\mathbf{V}$ contains a subdirectly irreducible lattice S of length n (by decreasing n if necessary). Then Lemma 14 shows us that a certain sublattice of $S \times S$ has an dense sublattice. □

Lemma 16 *Let $\mathbf{V}$ be a prevariety generated by lattices of width less than or equal to n, and suppose S is a subdirectly irreducible lattice in $\mathbf{V}$ of length at least $2n$. Let*

$$L = \{(x, y) \in S \times S \mid x \leq y\}$$

let $[a, b]$ be any critical quotient of S, and let L' be the sublattice of $S \times S$ generated by $L \cup \{(b, a)\}$. Then the inclusion of L in L' is an epimorphism in $\mathbf{V}$.

Proof: Let M be a lattice of width less than or equal to n, and let $f : L' \to M$ be a homomorphism. Let $x_0 < x_1 < \ldots < x_{2n}$ be a chain of length $2n$ in S; then

$$\{(x_0, x_{2n}), (x_1, x_{2n-1}), \ldots, (x_n, x_n)\}$$

is an antichain in L' with $n + 1$ elements. Therefore f is not injective. Suppose (p, q) and (r, s) are distinct elements such that $f((p, q)) = f((r, s))$. Taking joins and meets with $(0, 1)$, we get that $f((p, 1)) = f((r, 1))$, and $f((0, q)) = f((0, s))$. So the sublattices $\{0\} \times S$ and $S \times \{1\}$ are not both embedded in M by f. The proof is completed the same way as in Lemma 14. □

Theorem 17 *For any $n > 1$, let $\mathbf{L}^\infty_n$ be the variety generated by all lattices of width less than or equal to n. Then any subvariety of $\mathbf{L}^\infty_n$ has nonsurjective epimorphisms.*

Proof: Let $\mathbf{V}$ be a subvariety of $\mathbf{L}^\infty_n$. If there is a finite bound on the length of the subdirectly irreducible lattices in $\mathbf{V}$, then Theorem 15 applies to $\mathbf{V}$. Otherwise, let S be a subdirectly irreducible lattice in $\mathbf{V}$ of length at least $2n$; then Lemma 16 yields a nonsurjective epimorphism in $\mathbf{V}$. □

Lemma 18 *Let S_1 and S_2 be subdirectly irreducible lattices, not necessarily distinct. Let $[a,b]$ be a critical quotient of S_1, and let $[c,d]$ be a critical quotient of S_2. Let*

$$L = (S_1 \times [d)) \cup ((a] \times S_2) \subset S_1 \times S_2.$$

Then it is easy to see that L is a sublattice of $S_1 \times S_2$ that contains $\{0\} \times S_2$, $S_1 \times \{1\}$, (a,c), (a,d), and (b,d), but not (b,c). (More generally, we can let L be any sublattice that satisfies these conditions.) Let L' be the sublattice of $S_1 \times S_2$ generated by $L \cup \{(b,c)\}$. Let K be a class of lattices that includes S_1 and S_2. If there is no injective homomorphism from L' into any member of K, then the inclusion of L into L' is an epimorphism in $\mathbf{ISP}(K)$.

The proof is similar to those of Lemma 14 and Lemma 16, and is omitted.

Theorem 19 *Let S_1 and S_2 be splitting lattices, not necessarily distinct, and let $\mathbf{V_1}$ and $\mathbf{V_2}$ be their conjugate varieties. Let $\mathbf{V}$ be the variety generated by S_1, S_2, $\mathbf{V_1}$, and $\mathbf{V_2}$. Then there is an inclusion in $\mathrm{Var}(\{S_1, S_2\})$ that is an epimorphism in $\mathbf{V}$.*

To prove this, we need a lemma.

Lemma 20 *(Theorem 2.3(i) of [7]) Let $\mathbf{V}'$ and $\mathbf{V}''$ be two subvarieties of a congruence distributive variety. Then any subdirectly irreducible algebra in the variety generated by $\mathbf{V}'$ and $\mathbf{V}''$ is in $\mathbf{V}'$ or $\mathbf{V}''$.*

Proof of Theorem 19: Let $L \hookrightarrow L'$ be the inclusion given in Lemma 18. Let

$$K = \mathbf{V_1} \cup \mathbf{V_2} \cup \mathbf{HS}(S_1) \cup \mathbf{HS}(S_2).$$

By Lemma 20, $\mathbf{V}_{\mathrm{SI}} \subseteq K$, and therefore $\mathbf{V} = \mathbf{ISP}(K)$. L' has no embedding into any member of $\mathbf{V_1} \cup \mathbf{V_2}$ by the definition of conjugate variety, and it has no embedding into any member of $\mathbf{HS}(S_1) \cup \mathbf{HS}(S_2)$ because these lattices have fewer elements. Therefore the inclusion is an epimorphism. □

Cofinality of sublattices within their dominions

We call a subset S of a lattice L *cofinal* if for every $x \in L$, there is $y \in S$ such that $x \leq y$.

Theorem 21 *Let $\mathbf{V}$ be a variety such that every lattice in $\mathbf{P_U}(\mathbf{V}_{\mathrm{SI}})$ has the property that every sublattice is cofinal in its dominion. Then every lattice in $\mathbf{V}$ has this property.*

Proof: Let L be in $\mathbf{V}$. Then L can be expressed as a subdirect product of lattices $L_\alpha \in \mathbf{V}_{\mathrm{SI}}\,(\alpha \in I)$. Let p_α be the projection of $\prod_{\alpha\in I} L_\alpha$ onto L_α.

Suppose $L' \subseteq L$, and L' is not cofinal in $\mathrm{dom}_L^{\mathbf{V}}(L')$. Then there is some $z \in \mathrm{dom}_L^{\mathbf{V}}(L')$ such that there is no $x \in L'$ with $x \geq z$. Let

$$\mathcal{I} = \{S \in \mathbf{2}^I \mid \exists x \in L' \text{ such that } \forall \alpha \in S,\ f_\alpha(x) \geq f_\alpha(z)\}.$$

Clearly $\mathcal{I}$ is a proper ideal in $\mathbf{2}^I$. Therefore there is an ultrafilter over I that does not meet $\mathcal{I}$. Let L^* be the ultraproduct, and $p : \prod_{\alpha \in I} L_\alpha \to L^*$ the canonical map. Since L' dominates z, $p(L')$ dominates $p(z)$ in $p(L)$, hence in L^*. But there is no element of $p(L')$ greater than or equal to $p(z)$. Since $L^* \in \mathbf{P_U}(\mathbf{V}_{\mathrm{SI}})$ this is a contradiction. □

Now we will find some of the varieties to which this theorem applies.

Lemma 22 *For any lattice L, and $x_0, x_1 \in L$, the following are equivalent:*
a) There is a lattice M, and $x \in M$, such that there is an isomorphism $f : M[x] \to L$ with $(x, i) \mapsto x_i$, $(i = 0, 1)$.
b) x_0 is meet irreducible, x_1 is join irreducible, and x_1 covers x_0.

Proof: $(\Rightarrow)$ It is clear from the definition of $M[x]$ that $(x, 0)$ and $(x, 1)$ satisfy these properties, and the isomorphism preserves these properties.

$(\Leftarrow)$ Since x_0 is meet irreducible and x_1 is join irreducible, the interval $[x_0, x_1]$ is not perspective to any other interval of L. So $\mathrm{Con}(x_0, x_1)$ (the smallest congruence identifying x_0 and x_1) does not identify any other pair of elements. Let M be $L/\mathrm{Con}(x_0, x_1)$, let g be the canonical map $L \to M$, and let $x = g(x_0)$. Define $f : M[x] \to L$ by $(x, i) \mapsto x_i\, (i = 0, 1)$, and for all other $y \in M, y \mapsto g^{-1}(y)$. Then it is easy to see that the ordering of L is the same as that of $M[x]$. □

Lemma 23 *The class of lattices $D[d]$, where D is distributive and $d \in D$, is closed under ultraproducts.*

Proof: It suffices to show that this class is determined by first-order sentences. By Lemma 22 a lattice is of the form $L[d]$ if and only if it has elements x_0 and x_1 such that x_0 is meet irreducible, x_1 is join irreducible, and x_1 covers x_0. This property is clearly first-order. We can express the fact that L is distributive by saying that for any a, b, and c in $L[d]$, either

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

or

$$(a \vee b) \wedge c = x_1 \quad \text{and} \quad (a \wedge c) \vee (b \wedge c) = x_0$$

as this is the only failure of distributivity that would be corrected by γ. □

Theorem 24 *Every nontrivial variety of almost distributive lattices satisfies the hypothesis of Theorem 21.*

Proof: We know that in $\mathbf{D}$, every sublattice is cofinal in its dominion, so assume that $\mathbf{V}$ is larger than $\mathbf{D}$. Let $L \in \mathbf{P_U}(\mathbf{V}_{\mathrm{SI}})$. We know that $\mathbf{V}_{\mathrm{SI}}$ consists of lattices of the form $D[d]$. By Lemma 23, $\mathbf{P_U}(\mathbf{V}_{\mathrm{SI}})$ does also. So L is of the form $D[d]$. The remarks after Theorem 9 show that the only elements of $D[d]$ that can be nontrivially dominated are $(d, 0)$ and $(d, 1)$. If a sublattice M dominates $(d, 0)$, it must contain $(d, 1)$, so it is cofinal in its dominion. So suppose M dominates $(d, 1)$ and is not cofinal in its dominion. Then $(d, 0) \in M$. If there is $x \in M$, $x \not\leq (d, 1)$, then $x \vee (d, 0) > (d, 1)$, a contradiction. So $M \subseteq ((d, 0)]$. But note the map $f : D \to D$ given by $x \mapsto x \wedge d$ is a retraction of D onto $(d]$. Let g be the map from $(d] \subset D$ to $D[d]$ that maps d to $(d, 0)$ and maps every smaller element to itself. Then $gf\gamma$ is a retraction of $D[d]$ onto $((d, 0)]$. So M is contained in the dominion-closed sublattice $((d, 0)]$, a contradiction. $\square$

What we did for dominions, we can also do for epimorphisms.

Theorem 25 *Let $\mathbf{V}$ be a variety such that every lattice in $\mathbf{SP_U}(\mathbf{V}_{\mathrm{SI}})$ has the property that every dense sublattice is cofinal. Then every lattice in $\mathbf{V}$ has this property.*

Proof: Let L be in $\mathbf{V}$. Then L can be expressed as a subdirect product of lattices $L_\alpha \in \mathbf{V}_{\mathrm{SI}}\,(\alpha \in I)$. Let p_α be the projection of $\prod_{\alpha \in I} L_\alpha$ onto L_α. Suppose L' is an dense sublattice of L, and L' is not cofinal in L. Then there is some $z \in L$ such that there is no $x \in L'$ with $x \geq z$. Let

$$\mathcal{I} = \{S \in \mathbf{2}^I \mid \exists x \in L' \text{ such that } \forall \alpha \in S,\ f_\alpha(x) \geq f_\alpha(z)\}.$$

Clearly $\mathcal{I}$ is a proper ideal in $\mathbf{2}^I$. Therefore there is an ultrafilter over I that does not meet $\mathcal{I}$. Let L^* be the ultraproduct, and $p : \prod_{\alpha \in I} L_\alpha \to L^*$ the canonical map. Since L' is dense in L, $p(L')$ is dense in $p(L)$. But there is no element of $p(L')$ greater than or equal to $p(z)$. Since $p(L) \in \mathbf{SP_U}(\mathbf{V}_{\mathrm{SI}})$ this is a contradiction. $\square$

Note this proof is essentially the same as that of Theorem 21.

Theorem 25 has both a weaker hypothesis and a weaker conclusion then Theorem 21. So for the varieties to which Theorem 21 applies, it implies Theorem 25, but Theorem 25 applies to more varieties. In particular, it is not hard to see that Theorem 25 applies to $\mathrm{Var}(M_n)\,(n \geq 3)$ and $\mathrm{Var}(M_\omega)$. To show that Theorem 25 does not apply to every variety, we will use the following theorem:

Theorem 26 *(cf. Theorem 2.8 of [9]) Let S be a finite simple algebra such that $\mathrm{Var}(S)$ is congruence distributive, and let L be a subalgebra of S with more than one element. Then*

$$\mathrm{dom}_S^{\mathrm{Var}(S)}(L) = \{s \in S \mid \forall \phi \in \mathrm{Aut}(S), \phi|_L = id_L \Rightarrow \phi(s) = s\}.$$

Proof: Let f be a homomorphism from S to a subdirectly irreducible algebra $M \in \mathrm{Var}(S)$. Since S is finite, $M \in \mathbf{HS}(S)$, so if M is not isomorphic to S, then it has fewer elements. Therefore f either maps all of S to one element, or it is an isomorphism. If f and g are two such maps that agree on L but are not equal, then they are both isomorphisms, and their equalizer is the same as the equalizer of $g^{-1}f$ and id_S. □

This theorem implies that if L is a finite simple lattice with no nontrivial automorphisms, then in $\mathrm{Var}(L)$, any sublattice of L with more than one element is dense. Figure 1 shows an example of such a lattice:

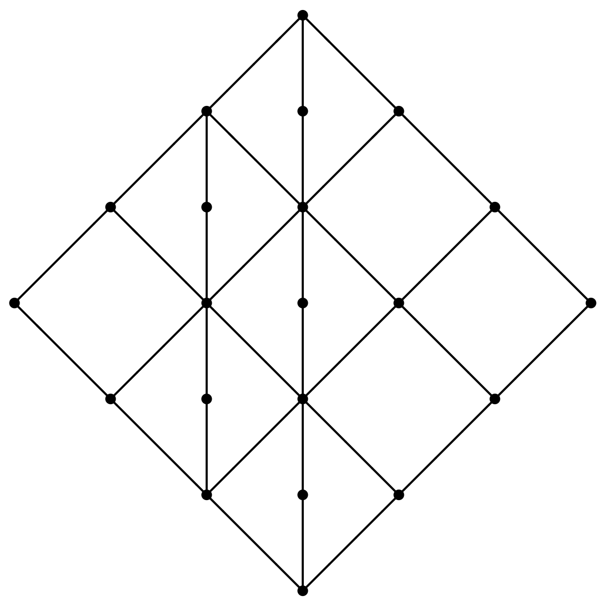


Figure 1: L

It is easy to see that if we collapse any edge of this lattice, then we must collapse one of the diamonds; since the diamonds are all linked, the whole lattice collapses. Therefore it is simple. To see that it has no nontrivial automorphisms, note that each element is uniquely determined by the set of elements that it covers and the number of elements that cover it. So by induction on the height of an element, any automorphism must fix every element.

Varieties closed under dominions

Corollary 2.12 of [8] shows that in any variety of groups, the dominion of an abelian subgroup is abelian. We would like to know if there is an analagous theorem for lattices. The following theorem shows that there is not.

Theorem 27 *Let* $\mathbf{V}$ *be a variety of lattices, such that for any variety of lattices* $\mathbf{W}$*, any* $L \in \mathbf{W}$*, and any sublattice* $L' \subseteq L$ *with* $L' \in \mathbf{V}$*,* $\mathrm{dom}_L^W(L') \in \mathbf{V}$*. Then* $\mathbf{V}$ *is* $\mathbf{L}$ *or* $\mathbf{T}$*.*

We will prove this by showing that dominions of distributive sublattices (in various varieties of lattices) generate all lattices. In particular, we will show that every finite partition lattice L (defined below) has an dense distributive sublattice in $\mathrm{Var}(L)$. This will prove the theorem, because Corollary IV.4.6 of [4] shows that the variety of lattices is generated by finite partition lattices.

For any set A, we let $\mathrm{Part}(A)$ be the set of all partitions of A, partially ordered by inclusion of the corresponding equivalence relations. Lemma IV.4.1 of [4] shows that this makes $\mathrm{Part}(A)$ a lattice, which we will call the *partition lattice* of A. Furthermore, Theorem IV.4.2 of [4] shows that $\mathrm{Part}(A)$ is simple and complete, and every element is a join of atoms. Note that $\mathrm{Part}(A)$ is finite if and only if A is.

We adopt the following notation: for any $S \subseteq A$, we write S^* for the partition of A such that S is the only block with more than one element.

Lemma 28 *Any bijection* $f : A \to A$ *induces an automorphism* ϕ_f *of* $\mathrm{Part}(A)$*, given by* $\pi \mapsto \{f(S) \,|\, S \in \pi\}$*. If* A *is finite, these are all the automorphisms of* $\mathrm{Part}(A)$*.*

Proof: It is trivial to show that ϕ_f is an automorphism. Now suppose A is finite; for convenience we will assume that $A = \{1, 2, \ldots, n\}$. The result is trivial for $n \leq 3$, so assume $n \geq 4$. Let ϕ be any automorphism of $\mathrm{Part}(A)$. Note that the atoms of $\mathrm{Part}(A)$ are the elements of the form $\{a, b\}^*$ with $a \neq b$, so ϕ must permute these. Also note that if $a, b, c, d \in A$ are distinct, then $\{a, b\}^* \vee \{a, c\}^*$ covers three atoms, but $\{a, b\}^* \vee \{c, d\}^*$ covers only two atoms. Therefore, if S and T are two-element subsets of A, let $U^* = \phi(S^*)$ and let $V^* = \phi(T^*)$; then U intersects V if and only if S intersects T. Now for $2 \leq i \leq n$, let $S_i^* = \phi(\{1, i\}^*)$. Then any two of the S_i's intersect. If $n > 4$, this already shows that they have an element in common. If $n = 4$, we must consider the case that they are $\{a, b\}, \{a, c\}$, and $\{b, c\}$, but this is impossible because

$$\{1, 2\}^* \vee \{1, 3\}^* \vee \{1, 4\}^* = 1_{\mathrm{Part}(A)}$$

but

$$\{a, b\}^* \vee \{a, c\}^* \vee \{b, c\}^* < 1_{\mathrm{Part}(A)}$$

So let $f(1)$ be the common element of the S_i's, and define the other values of f in the same way. Clearly f is bijective. Also, from the way f was constructed it is clear that ϕ agrees with ϕ_f on the atoms of $\mathrm{Part}(A)$; since every element of $\mathrm{Part}(A)$ is a join of atoms, $\phi = \phi_f$. □

This argument can be adapted to apply to the infinite case. In the last step we need the automorphism to respect infinite joins. But any automorphism of a lattice respects any infinite joins that exist, because it is also an automorphism of the partially ordered set.

Theorem 29 *Let* $A = \{1, 2, \ldots, n\}$. *Then there is a distributive sublattice* $L \subseteq \mathrm{Part}(A)$ *such that*

$$\mathrm{dom}_{\mathrm{Part}(A)}^{\mathrm{Var}(\mathrm{Part}(A))}(L) = \mathrm{Part}(A)$$

Proof: If n is 1 or 2, then $\mathrm{Part}(A)$ is already distributive, so assume $n \geq 3$. Let

$$L = \{0_{\mathrm{Part}(A)}, \{1,2\}^*, \{1,2,3\}^*, \ldots, \{1,2,\ldots,n\}^*\} \cup \{\{1,3\}^*\}.$$

Clearly L is distributive. Because $\mathrm{Part}(A)$ is finite and simple, by Theorem 26 it suffices to show that any automorphism of $\mathrm{Part}(A)$ that fixes every element of L is the identity. Let ϕ be such an automorphism; by Lemma 28, $\phi = \phi_f$ for some permutation f of A, and f must map each of the sets $\{1,2\}, \{1,3\}, \{1,2,3\}, \ldots$ into itself. Clearly f is the identity. □

Small sublattices with large dominions

For any finite or infinite cardinal κ, we would like to know how large the dominion of a κ-element sublattice can be. A one-element sublattice always has trivial dominion. Also, Corollary 1.5 of [5] shows that in any variety of algebras $\mathbf{V}$, if $A \in \mathbf{V}$ is infinite, and the number of operations of $\mathbf{V}$ is no greater than the cardinality of A, then for any $B \in \mathbf{V}$ with $A \subseteq B$, $\mathrm{dom}_B^{\mathbf{V}}(A)$ has the same cardinality as A. So if we are looking for small sublattices with large dominions, and we restrict the search to varieties, the best we can hope to find is a two-element sublattice with a countably infinite dominion. We will exhibit such a sublattice.

Let $\mathbf{M}_4^\infty$ be the variety generated by all modular lattices of width less than or equal to 4. We will say that $\{u, x, y, z, v\}$ *is a diamond* in L if these five elements form a sublattice isomorphic to M_3, u is the least element of this sublattice, and v is the greatest.

Let A_4 and A_7 be the lattices shown in the Figure 2. Let A_8 be the dual of A_7, and let A_9 be the projective plane over the two-element field.

Lemma 30 *(Corollary 3.3 of [2]) Let* L *be a modular lattice such that* $\mathbf{HS}(L)$ *does not contain* A_4, A_7, A_8, *or* A_9. *If* $\{u, x, y, z, v\}$ *and* $\{u, x, y', z', v'\}$ *are diamonds in* L, *then* $v = v'$.

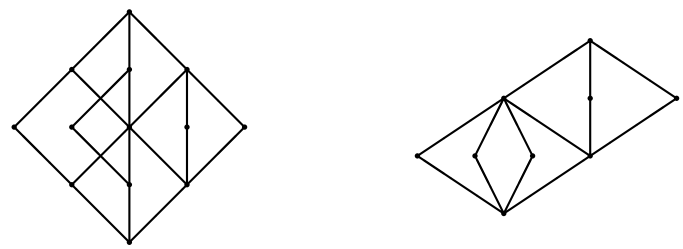


Figure 2: A_4 and A_7

Observe that A_4, A_7, A_8, and A_9 are all subdirectly irreducible lattices of width greater than 4. Therefore none of them is in $\mathbf{M}_4^\infty$, so none of them is in $\mathbf{HS}(L)$ for any $L \in \mathbf{M}_4^\infty$. This proves

Theorem 31 *Let $L = \{u, x, y, z, v\}$ be a diamond. Then*

$$\mathrm{dom}_L^{\mathbf{M}_4^\infty}(\{u, x\}) = \{u, x, v\}.$$

Now let B_∞ be the lattice in the Figure 3.

Theorem 32 $\mathrm{dom}_{B_\infty}^{\mathbf{M}_4^\infty}(\{u_1, u_2\}) = \{u_1, u_2, u_3, \ldots\}$.

Proof: We show that the u_n is in the dominion by induction on n: for $n \geq 3$, if u_{n-1} and u_{n-2} are in the dominion, then by Theorem 31, u_n is also. The opposite inclusion holds because there is an automorphism that moves all other elements of B_∞. □

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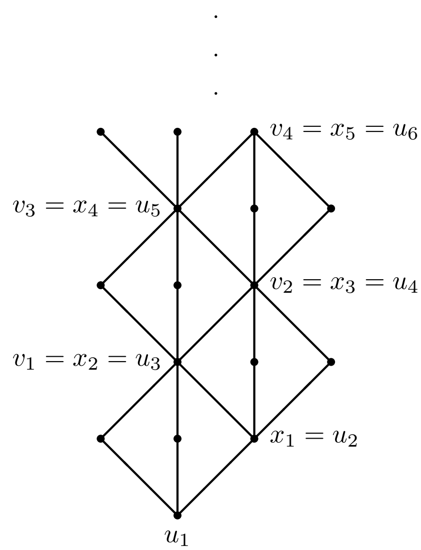

Figure 3: B_∞

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