

### Is there a nontrivial lattice that is not generated by the union of two proper sublattices?

I encountered this question when I was studying sublattice-lattices. Given a lattice  $L$ , let  $\text{Sub}(L)$  be the set of all sublattices of  $L$ , including  $\emptyset$ . Ordered by inclusion,  $\text{Sub}(L)$  is a lattice. It is clear that the only completely join irreducible members of  $\text{Sub}(L)$  are the one-element sublattices, but there may be other join irreducible elements.  $M \subseteq L$  is join irreducible in  $\text{Sub}(L)$  if and only if  $M$  is not the join of two proper sublattices of  $M$ , or equivalently, of finitely many proper sublattices.

I once guessed that every nontrivial lattice *is* the union of two proper sublattices. (For example, every lattice pictured in [1] has this property.) This is false; the simplest counterexample I know of is the projective plane over the three-element field.

Let us suppose that  $L$  is a nontrivial lattice that is not generated by the union of two proper sublattices, and see what this implies about  $L$ :

- (1)  $L$  does not have a minimal generating set. In particular, it is not finitely generated.
- (2) There is no homomorphism from  $L$  onto a nontrivial finite lattice. This implies that  $L$  is not in any finitely generated variety.
- (3)  $L$  has no maximal sublattice.

*Proof:* If  $L$  has a minimal generating set  $S$ , then since  $L$  is nontrivial,  $S$  has more than one element. So we can divide  $S$  into two disjoint proper subsets  $S_1$  and  $S_2$ . Since  $S$  is a minimal generating set, the sublattices generated by  $S_1$  and  $S_2$  are both proper. Their union contains  $S$ , so it generates  $L$ . This proves (1).

If  $M$  is a nontrivial finite lattice, and  $f : L \rightarrow M$  is a surjective homomorphism, then for each  $x \in M$ ,  $f^{-1}(x)$  is a proper sublattice of  $L$ , and the union of these finitely many sublattices is all of  $L$ , a contradiction. This proves (2).

If  $M$  is a maximal sublattice of  $L$ , then for any  $x \in L \setminus M$ ,  $\{x\} \cup M$  generates  $L$ . This proves (3).  $\square$

We can ask the same question about other kinds of algebras. I have answered this question for semilattices and groups:

For semilattices, the answer is no. Let  $L$  be a meet semilattice. Then there is some  $x \in L$  that is not the least element. Then  $[x]$  and  $L \setminus [x]$  are both proper subsemilattices.

For groups, the answer is yes. Let  $G$  be any finite cyclic group of prime-power order. If  $H_1$  and  $H_2$  are proper subgroups, then one is contained in the other, so their union does not generate  $G$ . This solution is trivial and uninteresting, so we should change the question to exclude it. Recall that in the lattice case, when we proved that  $L$  does not have a minimal generating set, we used the fact that a one-generated lattice is trivial. So perhaps “not one-generated” should be considered the proper analogue of “nontrivial.” With this restriction there is still a solution: the group  $\mathbf{Z}_{p^\infty}$ .

## References

- [1] G. Grätzer, *General Lattice Theory*, Second edition, Birkhäuser, Basel, 1998.