Small Kernels with Big Transfer Functions

Centered DFT, pseudoinverse MMSE, and truncated FIR

f_16-14.MCD, 12/19/94

$j := \sqrt{-1}$ $\quad N := 14 \quad i := 0..N-1 \quad k := 0..N-1$

$M := 7 \quad m := 0..M-1 \quad n := 0..M-1$

The 1-D Discrete Fourier Transform computes an M-point transfer function from an M-point impulse response:

$$F_n := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left(-j\cdot 2\cdot\pi\cdot n\cdot\frac{m}{M}\right) \quad \text{or} \quad F := W\cdot f \quad \text{where} \quad W_{n,m} := \frac{1}{\sqrt{M}}\cdot\exp\left(-j\cdot 2\cdot\pi\cdot n\cdot\frac{m}{M}\right)$$

is an M by M kernel matrix. Both [f] and [F] have the central element at the left end of the vector.

$f := (1 \;\; .5 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; .5)^T \qquad F := W\cdot f \qquad F^T = (0.76 \;\; 0.61 \;\; 0.29 \;\; 0.04 \;\; 0.04 \;\; 0.29 \;\; 0.61)$

Now let f(m) be an M - point FIR, rotated (a places) to the center of the vector.

$f := (0 \;\; 0 \;\; .5 \;\; 1 \;\; .5 \;\; 0 \;\; 0)^T \qquad a := \text{floor}\left(\frac{M}{2}\right) \quad a = 3 \qquad b := \text{floor}\left(\frac{N}{2}\right) \quad b = 7$

Its M-point transform is

$$F_n := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left(-j\cdot 2\cdot\pi\cdot n\cdot\frac{m-a}{M}\right) \qquad F^T = (0.76 \;\; 0.61 \;\; 0.29 \;\; 0.04 \;\; 0.04 \;\; 0.29 \;\; 0.61)$$

For the M-point transform, this works for any rotation, a, of the vector, for example, zero: $f := (1 \;\; .5 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; .5)^T$

$$F_n := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left(-j\cdot 2\cdot\pi\cdot n\cdot\frac{m-0}{M}\right) \qquad F^T = (0.76 \;\; 0.61 \;\; 0.29 \;\; 0.04 \;\; 0.04 \;\; 0.29 \;\; 0.61)$$

Now we can compute N points on its transform:

$$f := (0 \;\; 0 \;\; .5 \;\; 1 \;\; .5 \;\; 0 \;\; 0)^T \qquad G_k := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left(-j\cdot 2\cdot\pi\cdot k\cdot\frac{m-a}{N}\right)$$

$G^T = (0.76 \;\; 0.72 \;\; 0.61 \;\; 0.46 \;\; 0.29 \;\; 0.14 \;\; 0.04 \;\; 0 \;\; 0.04 \;\; 0.14 \;\; 0.29 \;\; 0.46 \;\; 0.61 \;\; 0.72)$

But for its N-point transform, the vector length is wrong and it doesn't wrap properly

$$G_k := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left(-j\cdot 2\cdot\pi\cdot k\cdot\frac{m-0}{N}\right)$$

$G^T = (0.76 \;\; 0.16-0.7j \;\; -0.55-0.27j \;\; -0.29+0.36j \;\; 0.18+0.23j \;\; 0.13-0.06j \;\; -0.01-0.04j \;\; 0 \;\; -0.01+0.04j \;\; 0.13+0.$

If the M-point function is centered on element a,
and we want its N-point spectrum to be centered on element b, we can use:

$f := (0 \;\; 0 \;\; .5 \;\; 1 \;\; .5 \;\; 0 \;\; 0)^T \qquad M = 7 \quad a = 3 \qquad N = 14 \quad b = 7$

so the transform is

$$G_k := \left(\frac{1}{\sqrt{M}}\right)\cdot\sum_{m=0}^{M-1} f_m\cdot\exp\left[-j\cdot 2\cdot\pi\cdot(k-b)\cdot\frac{m-a}{N}\right] \quad \text{or} \quad G := T\cdot f \quad \text{where} \quad T_{k,m} := \frac{1}{\sqrt{N}}\cdot\exp\left[-j\cdot 2\cdot\pi\cdot(k-b)\cdot\frac{m-a}{N}\right]$$

$G := T \cdot f$ or $\sqrt{\frac{N}{M}} \cdot G^T = (0\ \ 0.04\ \ 0.14\ \ 0.29\ \ 0.46\ \ 0.61\ \ 0.72\ \ 0.76\ \ 0.72\ \ 0.61\ \ 0.46\ \ 0.29\ \ 0.14\ \ 0.04)$

This scaling preserves the amplitude of the spectrum as it expands to N points.
This is then the general M-point to N-point centered DFT.

Small kernel from large MTF

Suppose the desired TF, given as an N-point spectrum vector centered on element b, is

$\sigma := 1.0$

$M = 7$

$$G_i := \exp\left[-\frac{(i-b)^2}{2 \cdot \sigma \cdot \sigma}\right] \qquad G^T = (0\ \ 0\ \ 0\ \ 0\ \ 0.01\ \ 0.14\ \ 0.61\ \ 1\ \ 0.61\ \ 0.14\ \ 0.01\ \ 0\ \ 0\ \ 0)$$

$N = 14$

whose N-point FIR is:

$$W_{i,k} := \frac{1}{\sqrt{N}} \cdot \exp\left[-j \cdot 2 \cdot \pi \cdot (i-b) \cdot \frac{k-b}{N}\right] \qquad g := \left(W^{-1} \cdot G\right)$$

$$g^T = (0.01\ \ 0.02\ \ 0.05\ \ 0.13\ \ 0.27\ \ 0.45\ \ 0.61\ \ 0.67\ \ 0.61\ \ 0.45\ \ 0.27\ \ 0.13\ \ 0.05\ \ 0.02)$$

We solve G = Tf (by the pseudoinverse method) for an M-point MMSE approximation FIR, f(m):

$$K := \left[\left(\left(\overline{T}\right)^T \cdot T\right)^{-1} \cdot \left(\overline{T}\right)^T\right] \qquad f := K \cdot G \qquad f^T = (0.27\ \ 0.45\ \ 0.61\ \ 0.67\ \ 0.61\ \ 0.45\ \ 0.27)$$

which, we observe, is simply the truncated FIR. The constant rescales T to preserve the amplitude of the FIR. We can transform this back to an N-point spectrum, to see the approximation TF:

$$F := T \cdot f \qquad F^T = (-0.05\ \ 0\ \ 0.05\ \ -0.02\ \ -0.05\ \ 0.2\ \ 0.65\ \ 0.89\ \ 0.65\ \ 0.2\ \ -0.05\ \ -0.02\ \ 0.05\ \ 0)$$

$$G^T = (0\ \ 0\ \ 0\ \ 0\ \ 0.01\ \ 0.14\ \ 0.61\ \ 1\ \ 0.61\ \ 0.14\ \ 0.01\ \ 0\ \ 0\ \ 0)$$

Using the N-point formulation as a check on the computation,

$$h := \text{augment}\left[(0\ \ 0\ \ 0\ \ 0), \text{augment}\left[f^T, (0\ \ 0\ \ 0)\right]\right]^T$$

$$h^T = (0\ \ 0\ \ 0\ \ 0\ \ 0.27\ \ 0.45\ \ 0.61\ \ 0.67\ \ 0.61\ \ 0.45\ \ 0.27\ \ 0\ \ 0\ \ 0)$$

$$H := W \cdot h \qquad H^T = (-0.05\ \ 0\ \ 0.05\ \ -0.02\ \ -0.05\ \ 0.2\ \ 0.65\ \ 0.89\ \ 0.65\ \ 0.2\ \ -0.05\ \ -0.02\ \ 0.05\ \ 0)$$

We can compute the MSE of the approximation $f^T = (0.27\ \ 0.45\ \ 0.61\ \ 0.67\ \ 0.61\ \ 0.45\ \ 0.27)$

$$g^T = (0.01\ \ 0.02\ \ 0.05\ \ 0.13\ \ 0.27\ \ 0.45\ \ 0.61\ \ 0.67\ \ 0.61\ \ 0.45\ \ 0.27\ \ 0.13\ \ 0.05\ \ 0.02)$$

$$RMSE := \sqrt{\frac{1}{N} \cdot \sum_{i=0}^{N-1} \left(F_i - G_i\right)^2}$$

$RMSE = 0.0550761$

Figure 16-14