

# Answers to Calculus Review, Differentiation

1.  $f(x) = \frac{4}{x}$

- a) Use the *definition* of the derivative to calculate the derivative of  $f$ .  
First, you need the definition! I'll write it here in terms of  $h$ , although we've also used  $\Delta x$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

It's important to substitute  $x + h$  correctly. After that, it's a bunch of algebra.

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h}$ . Now that's ugly, but it's just a complex fraction, and you can invariably clean those up by multiplying by the common denominator of the "little" fractions in both numerator and denominator.

$$\lim_{h \rightarrow 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} = \lim_{h \rightarrow 0} \frac{\frac{4x(x+h) - 4x(x+h)}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{h \cdot x(x+h)}$$
. Then distribute...

$$\lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{h \cdot x(x+h)} = \lim_{h \rightarrow 0} \frac{4x - 4x - 4h}{h \cdot x(x+h)} = \lim_{h \rightarrow 0} \frac{-4h}{h \cdot x(x+h)} = \lim_{h \rightarrow 0} \frac{-4}{x(x+h)}$$
. Now that the  $h$  term is out

of the denominator, there should be no trouble with letting  $h$  equal zero. Here goes.

$$\lim_{h \rightarrow 0} \frac{-4}{x(x+h)} = \frac{-4}{x \cdot x} = \frac{-4}{x^2} = f'(x)$$

Note that  $\frac{4}{x} = 4x^{-1}$ . Then  $\frac{d}{dx}[4x^{-1}] = -4x^{-2} = -\frac{4}{x^2}$ . It's a whole lot easier. (Remember now

why you loved the power rule at first sight?) But that process doesn't use the *definition* of derivative, and would therefore not be a correct answer to the question I asked.

The question of how complicated I'm likely to make something like this on the test is completely legitimate. The answer is, "not very." Which is to say that **I'm far more likely to ask you how to set up such a limit than I am to have you do all of the tedious algebra involved in evaluating it**. I don't think that this algebra would be the best use of the time I have to evaluate your understanding of differentiation. But how to *use* the definition with any function I might throw at you — that appeals to me.

- b) Write an equation of the tangent line to the graph of  $f$  at the point  $(-2, -2)$ .

To get a tangent line, I need (or, rather, *you* need) a slope and a point. Hey, look, there's a point right in the question! To find the slope of a tangent line, a/k/a the slope of a curve at a point, you use the derivative. Therefore slope  $= f'(-2) = -\frac{4}{(-2)^2} = -1$

And with that information, I can write an equation for the tangent. I'll use point-slope, partly because I'm fond of it, mostly because I'm lazy, and coincidentally because it will make the next part of the question really easy.

$$y + 2 = -1(x + 2)$$

- c) Write an equation of the normal line to the graph for  $f$  at the point  $(-2, -2)$ .

The slope of the normal is 1, because the normal is perpendicular to the tangent.

$$y + 2 = 1(x + 2)$$

2. a) DNE; there's a sharp corner at  $a$ . e) positive; at  $e$  (on the axis),  $g$  slopes up  
 b) 0; the tangent line at  $b$  is horizontal f) DNE; the function isn't continuous at  $f$   
 c) negative; at  $c$ , the graph slopes down g) DNE; there's a vertical tangent, and the slope of a vertical line is undefined.  
 d) 0; at  $d$ , there's a horizontal tangent

Now for 33 consecutive “differentiate” questions (not including logarithmic differentiation, which is the next section of problems).

3.  $y = x\sqrt[3]{x+1}$

First, rewrite the radical as a power, then use the product rule.

$$y = x\sqrt[3]{x+1} = x(x+1)^{1/3}$$

Using the product rule...

$$y' = x \cdot \frac{1}{3}(x+1)^{-2/3} \cdot 1 + (x+1)^{1/3} \cdot 1$$

$$= \frac{1}{3}x(x+1)^{-2/3} + (x+1)^{1/3}$$

That’s okay, and you could stop there, but I’m going to do the fancy algebra thing and factor out both the lowest power of  $(x+1)$ , which is the  $-\frac{2}{3}$  power, and the constant coefficient,  $\frac{1}{3}$ , because it’s sort of ugly.

$$y' = \frac{1}{3}(x+1)^{-2/3}(x+3(x+1))$$

“Where did the 3 in the parentheses come from?” If I were to multiply this back through, I’d need to get one as the coefficient of the second part. and  $\frac{1}{3} \cdot 3 = 1$ . “Okay, but how did you get just plain  $(x+1)$  with it?”<sup>‡</sup> It’s the same thing, really:  $(x+1)^1 \cdot (x+1)^{-2/3}$  gives  $(x+1)^{1/3}$ , which was what I started with. More mathematically, factoring is dividing by the factor. If I divide  $(x+1)^{1/3}$  by  $(x+1)^{-2/3}$ , I subtract the exponents, and  $\frac{1}{3} - (-\frac{2}{3}) = 1$ . Time for more simplifying.

$$y' = \frac{1}{3}(x+1)^{-2/3}(x+3x+3)$$

$$= \frac{1}{3}(x+1)^{-2/3}(4x+3)$$

$$y' = \frac{4x+3}{3(x+1)^{2/3}} = \frac{4x+3}{3\sqrt[3]{(x+1)^2}}$$

Remember that in a multiple choice question, you have to be able to match the format of the answer (or at least figure out which answer matches yours), so occasionally doing the algebra is good practice.

<sup>‡</sup> If you find the question and answer format here a little strange, you should know that it’s how Plato wrote. And he’s a famous philosopher. He wrote all of his stuff as dialogues between Socrates (his teacher) and an endless supply of people with questions. So there. And the cool looking symbol I used for the footnote is called a “double-dagger.”

4.  $f(x) = (2x^2 + 5)^7$   
 🎵 “Chain, chain, chain, ...” 🎵  
 $f'(x) = 7(2x^2 + 5)^6 \cdot 4x$   
 $f'(x) = 28x(2x^2 + 5)^6$

That was nice.

5.  $f(x) = (x^2 + 1)(x^3 + 3)$

I have a couple of choices here. I could use the product rule, or I could multiply the binomials (you know, FOIL them) and just use the power rule. I’m going with FOIL.

$$f(x) = x^5 + x^3 + 3x^2 + 3$$

$$f'(x) = 5x^4 + 3x^2 + 6x$$

6.  $y = \left(\frac{x}{x^2+1}\right)^{-1}$

I could use the power rule right away, and the chain rule would have me use the quotient rule to get the derivative of the inside. I *could* do that, but I won’t. “Why not?” Because the  $-1$  power just means that I need to (all together now...) *Flip That Sucker Over!* Watch.

$$y = \left(\frac{x}{x^2+1}\right)^{-1} = \frac{x^2+1}{x}$$

Isn’t that better? No chain rule, just a quotient. However, because I’m the math teacher, I’m going to be even sneakier and break that fraction into two pieces. If I divide each of the terms in the numerator by  $x$ , I wind up with  $y = x + x^{-1}$ , and all I’ll need is the power rule.

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

So you can see that it’s the same, I’ll work out the chain-quotient combo, too.

$$\frac{dy}{dx} = -1 \left(\frac{x}{x^2+1}\right)^{-2} \cdot \frac{(x^2+1) \cdot 1 - x \cdot 2x}{(x^2+1)^2}$$

$$= -1 \cdot \frac{\cancel{(x^2+1)}^2}{x^2} \cdot \frac{x^2+1-2x^2}{\cancel{(x^2+1)}^2}$$

$$= -1 \cdot \frac{-x^2+1}{x^2} = \frac{x^2-1}{x^2}$$

And there you go.

$$7. \quad r(x) = 3x^2 - \frac{2}{x} + \frac{5}{x^2}$$

$$= 3x^2 - 2x^{-1} + 5x^{-2}$$

I made it into powers to use the power rule.

$$r'(x) = 6x + 2x^{-2} - 10x^{-3}$$

$$r'(x) = 6x + \frac{2}{x^2} - \frac{10}{x^3}$$

And I put it back into fractions to match the original format.

$$8. \quad y = \frac{\pi}{x}$$

You could do this problem with the quotient rule, but it's overkill, because  $\pi$  is just a constant. If you rewrite  $y$  as a power, you get a simpler function to differentiate.

$$y = \frac{\pi}{x} = \pi x^{-1}$$

$$\frac{dy}{dx} = -\pi x^{-2} = -\frac{\pi}{x^2}$$

$$9. \quad y = \frac{x}{\pi}$$

The similarity to the previous question is intentional. The  $\pi$  is *still* a constant, and you *still* don't need the quotient rule. If, instead of writing it as a division, I write it as a multiplication, it's really simple.

$$y = \frac{x}{\pi} = \frac{1}{\pi} \cdot x$$

$$\frac{dy}{dx} = \frac{1}{\pi}$$

$$10. \quad y = 1 + \sqrt{2} \csc x + \cot x$$

This one is very straightforward, provided you still remember the derivatives of all six trig functions.

$$y' = 0 + \sqrt{2} \cdot -\csc x \cot x - \csc^2 x$$

$$y' = -\sqrt{2} \csc x \cot x - \csc^2 x, \text{ or, factoring,}$$

$$y' = -\csc x (\sqrt{2} \cot x + \csc x)$$

$$11. \quad g(x) = \tan(5 - \sin 2x)$$

Use the chain rule.

$$g'(x) = \sec^2(5 - \sin 2x) \cdot -\cos 2x \cdot 2$$

Did you catch that extra 2 on the end?

There's a second "inside" in the sine function; I multiplied by its derivative, too.

$$g'(x) = -2 \cos 2x \sec^2(5 - \sin 2x)$$

$$12. \quad y = 5 \cot\left(\frac{2}{x}\right) = 5 \cot(2x^{-1})$$

I hear Aretha calling out to me!

$$y' = 5 \cdot -\csc^2(2x^{-1}) \cdot -2x^{-2} = \frac{10}{x^2} \csc^2\left(\frac{2}{x}\right)$$

Some of you might be wondering if you can turn the cosecant of that ugly fraction into the sine of its reciprocal. The answer is no. The argument of cosecant is an *angle*, not a ratio, and the reciprocal of an angle doesn't make much sense mathematically.

$$13. \quad y = \sin(x + y)$$

Since you can't isolate  $y$ , use implicit differentiation. I'm going to use  $y'$  for the derivative of  $y$ . I could use  $\frac{dy}{dx}$ . It's all the

same.

$$y' = \cos(x + y) (1 + y')$$

Distribute.

$$y' = \cos(x + y) + y' \cos(x + y)$$

Collect  $y'$  terms on the left.

$$y' - y' \cos(x + y) = \cos(x + y)$$

Factor out that  $y'$ .

$$y'(1 - \cos(x + y)) = \cos(x + y)$$

And divide.

$$y' = \frac{\cos(x + y)}{1 - \cos(x + y)}$$

$$14. \quad g(x) = \frac{1}{\sqrt[4]{6 - x^3}} = (6 - x^3)^{-1/4}$$

The quotient rule is overkill when the numerator is a constant, since its derivative is 0. It's much more sensible to write those as powers instead.

$$g'(x) = -\frac{1}{4}(6 - x^3)^{-5/4} \cdot -3x^2$$

$$g'(x) = \frac{3x^2}{4(6 - x^3)^{5/4}}$$

$$15. \quad y^2 = x$$

I could solve for  $y$ , but it would be harder to differentiate, and it would involve a  $\pm$ , and I just don't need that. So it's implicit for me...

$$2y y' = 1$$

$$y' = \frac{1}{2y}$$

16.  $x^2 - xy + y^2 = 7$   
Clearly this one is implicit, too.

Differentiating,

$$2x - x \cdot y' - y \cdot 1 + 2y \cdot y' = 0$$

$$-xy' + 2yy' = y - 2x$$

$$y'(2y - x) = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

If you had moved the  $y'$  terms to the right side, instead, you'd have ended up with  $\frac{2x - y}{x - 2y}$ , which is the same thing. Notice the symmetry, too.

17.  $y = x\sqrt{x^2 + 1} = x(x^2 + 1)^{1/2}$

Using the product rule,

$$\frac{dy}{dx} = x \cdot \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x + (x^2 + 1)^{1/2} \cdot 1$$

$$= \frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1}$$

And that's fine. But here's some fancy factoring like I did back in problem 3.

$$= (x^2 + 1)^{-1/2} (x^2 + (x^2 + 1)^1)$$

$$\frac{dy}{dx} = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

18.  $y = \sqrt{1 - \sqrt{x}} = (1 - x^{1/2})^{1/2}$

*U-G-L-Y, you ain't got no alibi...* — uh, never mind. It's just the chain rule, and it's actually much prettier with powers than it was with roots.

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^{1/2})^{-1/2} \cdot -\frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dx} = -\frac{1}{4}(1 - x^{1/2})^{-1/2} x^{-1/2} = \frac{-1}{4\sqrt{x(1 - \sqrt{x})}}$$

19.  $y^4 = y^2 - x^2$

Implicit differentiation:

$$4y^3 y' = 2yy' - 2x$$

Remember that it's only the terms with  $y$  in them that need the  $y'$  on the end. Collect  $y'$  terms on one side.

$$4y^3 y' - 2yy' = -2x$$

Factor out the  $y'$ .

$$y'(4y^3 - 2y) = -2x$$

Divide.

$$y' = \frac{-2x}{4y^3 - 2y} = \frac{-2x}{2(2y^3 - y)} = \frac{-x}{2y^3 - y}$$

20.  $x^2 \cos^2 y - \sin y = 0$ , which means

$$x^2 (\cos y)^2 - \sin y = 0.$$

Differentiating with the product rule,

$$x^2 \cdot 2(\cos y)^1 \cdot -\sin y \cdot y' + (\cos y)^2 \cdot 2x$$

$$- \cos y \cdot y' = 0$$

$$-2x^2 \cos y \sin y \cdot y' + 2x \cos^2 y - \cos y \cdot y' = 0$$

$$2x \cos^2 y = 2x^2 \cos y \sin y \cdot y' + \cos y \cdot y'$$

$$2x \cos^2 y = y'(2x^2 \cos y \sin y + \cos y)$$

$$y' = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y}$$

$$= \frac{2x \cos y \cdot \cos y}{\cos y (2x^2 \sin y + 1)} = \frac{2x \cos y}{2x^2 \sin y + 1}$$

Yes, it would be nice to cancel the 2's, but I can't do it. It's not a factor of the entire denominator.

21.  $f(x) = x \ln x$

Product rule...

$$f'(x) = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$$

22.  $y = \ln\left(\frac{\sqrt{4 + x^2}}{x}\right)$

This is not the prettiest question. You could certainly work it as it stands, but the fraction and radical are not a great combination. So let's Harness the Power of the Logarithm<sup>®</sup>.

$$y = \ln\left(\frac{\sqrt{4 + x^2}}{x}\right) = \frac{1}{2} \ln(4 + x^2) - \ln x$$

That's better!

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + x^2} \cdot 2x - \frac{1}{x} = \frac{x}{4 + x^2} - \frac{1}{x}$$

or, if you're obsessive,

$$= \frac{x^2 - (4 + x^2)}{x(4 + x^2)} = \frac{-4}{x(4 + x^2)}$$

If you're wondering why I didn't do that part where you multiply by  $y$  at the end, it's because this isn't really the process called "logarithmic differentiation." That's when you have something so hideous that you actually take the logarithms of both sides of the original formula for  $y$ . In that case, differentiating the left side gives an extra  $y$ . You'll see it starting with question 36.

<sup>®</sup> Actually, this isn't meant to be a footnote; I'm marking the phrase as my "registered" trademark. I'm registering it with you, right now.

$$23. \quad y = \ln\left(\frac{1}{x}\right)$$

Yes, I could take the derivative of this as-is, but I'm way too lazy for that.

$$y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x. \text{ Ah, yes!}$$

$$\frac{dy}{dx} = -\frac{1}{x}$$

$$24. \quad g(x) = \ln|\sec x + \tan x|$$

There are no fancy tricks to make this one better. It's just the chain rule.

$$g'(x) = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

That result, on the other hand, cleans up quite nicely.

$$= \frac{\sec x (\cancel{\tan x} + \sec x)}{\cancel{\sec x} + \tan x} = \sec x$$

$$25. \quad y = \ln(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

Nice.

$$26. \quad y = (\ln x)^5$$

♪ That's the sound of the men working on the chain ga-a-ang, oh, don't you know...♪

$$\frac{dy}{dx} = 5(\ln x)^4 \cdot \frac{1}{x} = \frac{5(\ln x)^4}{x}$$

$$27. \quad g(x) = \arctan x$$

$$g'(x) = \frac{1}{1+x^2}$$

(You memorized this one, right? You really need to memorize all of those. You've been warned.)

$$28. \quad h(x) = \operatorname{arccsc}(e^x)$$

Use the chain rule. The derivative of  $\operatorname{arccsc} x$

is  $\frac{-1}{|x|\sqrt{x^2-1}}$ , if you've forgotten.

$$h'(x) = \frac{-1}{|e^x|\sqrt{(e^x)^2-1}} \cdot e^x = \frac{-e^x}{e^x \sqrt{e^{2x}-1}}$$

$$= \frac{-1}{\sqrt{e^{2x}-1}}$$

Note that because  $e^x$  is always positive, I was able to eliminate the absolute value and reduce.

$$29. \quad y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

It's a quotient rule. It will turn out fairly well, but the intermediate steps are not so nice.

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

The numerator simplifies. Watch.

$$= \frac{(e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{\cancel{e^{2x}} + 2e^x e^{-x} + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2e^x e^{-x} - \cancel{e^{-2x}}}{(e^x + e^{-x})^2}$$

$$= \frac{4e^x e^{-x}}{(e^x + e^{-x})^2} = \frac{4e^0}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

Did you catch that slick add-the-exponents thing? Niiiiice.

$$30. \quad f(x) = \log_a a^{\sin x}$$

That simplifies. Logarithms and exponential functions are inverses of each other.

$$f(x) = \log_a a^{\sin x} = \sin x$$

$$f'(x) = \cos x$$

$$31. \quad y = e^{\sqrt{x}}$$

Just a nice simple chain rule.

$$\frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2}$$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$32. \quad y = x^3 3^x$$

Product rule. Remember how to take the derivative of an exponential function with a base other than  $e$ ? It's just like  $e^x$ , but multiplied by the natural logarithm of the base.

$$y' = x^3 \cdot 3^x \ln 3 + 3^x \cdot 3x^2$$

$$y' = x^2 3^x (x \ln 3 + 3)$$

$$33. \quad g(x) = \ln\left(\frac{10}{x}\right)$$

HtPotL®!

$$g(x) = \ln\left(\frac{10}{x}\right) = \ln 10 - \ln x$$

$$g'(x) = -\frac{1}{x}$$

$$34. \quad y = \ln\left(\frac{\sqrt{x}}{5-x}\right)$$

More harnessing. Let's go.

$$y = \ln\left(\frac{\sqrt{x}}{5-x}\right) = \frac{1}{2}\ln x - \ln(5-x)$$

Now take derivatives.

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x} - \frac{1}{5-x} \cdot -1$$

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{1}{5-x}$$

$$35. \quad y = \log_4(4 + x \ln 4)$$

The derivative of a log of a base other than  $e$  has the natural log of the base in the denominator.

$$\frac{dy}{dx} = \frac{1}{(4 + x \ln 4) \ln 4} \cdot \ln 4 = \frac{1}{4 + x \ln 4}$$

Hey, that was the end of the huge page full of derivatives! Now there are... more derivatives. But this time it's logarithmic differentiation, which is more interesting. It leads to this question:

36. What is logarithmic differentiation, and why is it used?

The idea behind logarithmic differentiation is that you have a function to differentiate that is either (a) so ugly even it's mother won't acknowledge it or (b) has variables in both the base and the exponent. In each case, one Harnesses the Power of the Logarithm to simplify the function to something manageable before differentiating. You are taking natural logs of both sides of a problem that didn't have any to begin with.

$$37. \quad y = \frac{\sqrt{3x+1}}{(x-2)\sqrt[4]{x+8}}$$

Logarithmitize (if it wasn't a word before, it is now).

$$\ln y = \ln\left(\frac{\sqrt{3x+1}}{(x-2)\sqrt[4]{x+8}}\right)$$

Now break down the ugly as much as possible.

$$\ln y = \frac{1}{2}\ln(3x+1) - \ln(x-2) - \frac{1}{4}\ln(x+8)$$

Notice how there's a negative on the last term, too. It's divided, too.

Now differentiate both sides. The left side will need a touch of implicit differentiation, too.

$$\frac{1}{y} \cdot y' = \frac{1}{2} \cdot \frac{1}{3x+1} \cdot 3 - \frac{1}{x-2} \cdot 1 - \frac{1}{4} \cdot \frac{1}{x+8} \cdot 1$$

$$\frac{y'}{y} = \frac{3}{6x+2} - \frac{1}{x-2} - \frac{1}{4x+32}$$

And that's pretty nice. But it's not done. In order to solve for  $y'$ , I'll multiply both sides by the original  $y$  — which is the big radical-infested fraction. So here's the result.

$$y' = \left(\frac{3}{6x+2} - \frac{1}{x-2} - \frac{1}{4x+32}\right) \frac{\sqrt{3x+1}}{(x-2)\sqrt[4]{x+8}}$$

$$38. \quad y = x^{\cos x}$$

Taking natural logarithms of both sides gives

$$\ln y = \ln x^{\cos x} = \cos x \ln x$$

Now I'll differentiate both sides, using the product rule on the right side.

$$\frac{y'}{y} = \cos x \cdot \frac{1}{x} + \ln x \cdot -\sin x$$

$$\frac{y'}{y} = \frac{\cos x}{x} - \ln x \sin x$$

And now I can solve for  $y'$  by multiplying both sides by  $y$  (a/k/a  $x^{\cos x}$ ).

$$y' = \left(\frac{\cos x}{x} - \ln x \sin x\right) x^{\cos x}$$

$$39. \quad y = (x+1)^{x+1}$$

Same deal. I need to get the exponent to be not-an-exponent, and logarithms are the way to do that.

$$\ln y = \ln(x+1)^{x+1} = (x+1) \ln(x+1)$$

Now differentiate both sides.

$$\frac{y'}{y} = (x+1) \cdot \frac{1}{x+1} \cdot 1 + \ln(x+1) \cdot 1$$

$$\frac{y'}{y} = 1 + \ln(x+1)$$

Aaaand multiply by the original  $y$ .

$$y' = [1 + \ln(x+1)](x+1)^{x+1}$$

$$40. \quad y = x^{\frac{1}{\ln x}}$$

Yeah, yeah, whatever.

$$\ln y = \frac{1}{\ln x} \cdot \ln x = 1$$

Wait. That was weird. Differentiate.

$$\frac{y'}{y} = 0$$

$$y' = 0$$

But if the derivative of a function is 0, the function must be a horizontal line. In fact, since  $\ln y = 1$ ,  $y = e$ . But  $y = x^{\frac{1}{\ln x}}$ . So  $e = x^{\frac{1}{\ln x}}$ . If this seems too weird, try graphing the original function. It's true.

41. Find  $\frac{d^2y}{dx^2}$  for  $y = \frac{x+2}{x-3}$ .

Use the quotient rule. That notation means the *second* derivative. But to find the second derivative, first you have to find the first derivative. And, no, that wasn't redundant.

$$y' = \frac{(x-3) \cdot 1 - (x+2) \cdot 1}{(x-3)^2} = \frac{x-3-x-2}{(x-3)^2} = \frac{-5}{(x-3)^2} = -5(x-3)^{-2}$$

In general, I probably would have left the fraction, but I need to take another derivative. By writing this as a power, I can just use the chain rule!

$$y'' = -5 \cdot -2(x-3)^{-3} = 10(x-3)^{-3} = \frac{10}{(x-3)^3}$$

42. Find  $\frac{d^2y}{dx^2}$  for  $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 4$ .

Differentiating,

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$

Ugh. Multiplying both sides of the equation by 3 will remove the denominators.

$$3 \cdot \left( \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' \right) = 0 \cdot 3$$

$$x^{-2/3} + y^{-2/3}y' = 0$$

"Hey, can we raise both sides to the  $-3/2$  power to get rid of the exponents?" Great idea! But no, it won't work. First off, it would *also* raise the  $y'$  term to a power, which is unorthodox. Unorthodox isn't necessarily wrong. However, the  $-3/2$  power of a binomial doesn't distribute the way you'd like it to. Think about how squaring a binomial goes; you'd need FOIL, and there's a middle term. It's even worse with a fractional power. I'm pretty sure there's nothing you could do

that would fix it. So you just have to work with what you've got. Subtract the  $x$  term, and divide by the coefficient of  $y'$ .

$$y^{-2/3}y' = -x^{-2/3}$$

$$y' = \frac{-x^{-2/3}}{y^{-2/3}} = -\frac{y^{2/3}}{x^{2/3}} = -\left(\frac{y}{x}\right)^{2/3}$$

Did you catch all of the algebraic virtuosity there? The negative powers let the terms move to the opposite positions in the fraction, and thinking about it as a single power will, I think, make the second derivative far less annoying. That's a matter of personal preference, though. As I've written it, with the power on the parentheses, I'll have to use the chain rule, which I wouldn't have needed before. However, in the quotient rule, for the inside function, I won't have fractional powers. It's my problem, and this is the way I like it.

$$y'' = -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3} \cdot \frac{xy' - y \cdot 1}{x^2}$$

It's not bad, but you don't want to leave the  $y'$  there. It would mean an extra computation to get the second derivative at a point. (In the next chapter, you'll learn what the second derivative tells you about a function.) So, since I already know what  $y'$  is, I'll plug it in.

$$y'' = -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3} \cdot \frac{x \cdot -\left(\frac{y}{x}\right)^{2/3} - y}{x^2}$$

*That's ugly.* I think you should simplify *some*, probably to the first boxed answer below. I continued by factoring, and I'm giving you that answer, too, but I really think I'd stop at the first box below under normal circumstances.

$$y'' = -\frac{2}{3} \cdot \frac{y^{-1/3}}{x^{-1/3}} \cdot \frac{x \cdot -y^{2/3}x^{-2/3} - y}{x^2}$$

$$= -\frac{2x^{1/3}}{3y^{1/3}} \cdot \frac{-x^{-1/3}y^{2/3} - y}{x^2}$$

$$= \frac{2y^{2/3} + 2x^{1/3}y}{3y^{1/3}x^2}$$

Here comes the masochism, or, if you prefer, the skills that make me qualified to write the answers in the back of a textbook. I'll factor out a  $y^{1/3}$  in the numerator so I can reduce the fraction.

$$y'' = \frac{y^{1/3}(2y^{1/3} + 2x^{1/3}y^{2/3})}{3y^{1/3}x^2}$$

$$y'' = \frac{2y^{1/3} + 2x^{1/3}y^{2/3}}{3x^2}$$

If I *really* wanted to go all out, I'd factor out the  $2y^{1/3}$  from the numerator, but it doesn't actually make anything *better*, so I won't. ©

43. Let  $f(3) = 0$ ,  $f'(3) = 6$ ,  $g(3) = 1$ , and  $g'(3) = \frac{1}{3}$ . Find stuff, if possible.

a)  $h'(3)$  if  $h(x) = f(x)g(x)$   
Use the product rule.

$$h'(x) = f(x)g'(x) + g(x)f'(x).$$

$$\text{So } h'(3) = 0 \cdot \frac{1}{3} + 1 \cdot 6 = 6$$

b)  $j'(3)$ , if  $j(x) = g(f(x))$

This time it's the chain rule.

$$j'(x) = g'(f(x))f'(x)$$

$$j'(3) = g'(f(3)) \cdot 6 = g'(0) \cdot 6 = 6g'(0)$$

Unfortunately, I don't have any information about  $g'(0)$ , so I'm stuck there.

c)  $k'(3)$ , if  $k(x) = \frac{g(x)}{h(x)}$

Lo-d-hi! I mean, quotient rule!

$$k'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

$$k'(3) = \frac{h(3)g'(3) - g(3)h'(3)}{(h(3))^2}$$

Ummm...  $h(3)$  isn't in the list.

Waitwaitwait! That function was in part (a)! So  $h(3) = f(3)g(3) = 0 \cdot 1 = 0$ , and  $h'(3) = 6$ , which was what I found in (a). Back to  $k'$ .

$$k'(3) = \frac{0 \cdot \frac{1}{3} - 1 \cdot 6}{0^2} = \frac{-6}{0}$$

So  $k'(3)$  doesn't exist. But  $k$  *does* have a vertical tangent when  $x = 3$ , so I did get some information out of this. (BTW, did you like my simulation of excitement in the previous paragraph?)

d)  $m'(3)$ , if  $m(x) = 3f(x) - 2g(x)$

If you need a name for it, this is the constant multiple rule.

$$m'(x) = 3f'(x) - 2g'(x)$$

$$\begin{aligned} m'(3) &= 3f'(3) - 2g'(3) \\ &= 3 \cdot 6 - 2 \cdot \frac{1}{3} = 17\frac{1}{3} \end{aligned}$$

44. A particle starts at time  $t = 0$  and moves along the  $x$ -axis so that its position at any time  $t \geq 0$  is given by  $x(t) = (t - 1)^3(2t - 3)$ .

a) Find the displacement of the particle from  $t = 2$  to  $t = 5$ .

Displacement is change in position.

Position at time 2 is

$$x(2) = (2 - 1)^3(2 \cdot 2 - 3) = 1 \cdot 1 = 1.$$

Position at time 5 is

$$x(5) = (5 - 1)^3(2 \cdot 5 - 3) = 4^3 \cdot 7 = 448.$$

$$\text{So displacement is } 448 - 1 = 447.$$

No units in the problem means no units in the answer.

b) Find the average velocity of the particle from  $t = 2$  to  $t = 5$ .

Average velocity is change in position divided by change in time. Change in position is the displacement, which I just found, and change in time is 3.

$$\text{So } 447 \div 3 = 149.$$

c) Find a formula for the instantaneous velocity of the particle at any time  $t \geq 0$ .  $v(t) = x'(t)$ , which means use of the product rule.

$$\begin{aligned} v(t) &= (t - 1)^3 \cdot 2 + (2t - 3) \cdot 3(t - 1)^2 \cdot 1 \\ &= (t - 1)^2(2t - 2 + 6t - 9) \\ &= (t - 1)^2(8t - 11) \end{aligned}$$

Yes, I factored in the second line. You could have left it as

$$v(t) = 2(t - 1)^3 + 3(2t - 3)(t - 1)^2,$$

but I looked ahead, and I'm going to need a second derivative. The nicer it looks at this step, the fewer opportunities I have to make mistakes.

d) For what values of  $t$  is the velocity of the particle less than zero?

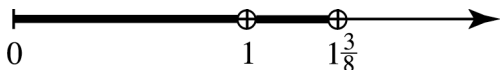
$$v(t) < 0, \text{ so } (t - 1)^2(8t - 11) < 0$$

The zeros of  $(t - 1)^2(8t - 11)$  are  $t = 1$  and

$$t = \frac{11}{8} = 1\frac{3}{8}.$$

The simplest way to solve the inequality is to use a number line and test points. Do you remember this from back in the day? The domain starts at 0, and I'll mark circles at the zeros of velocity, since I'm only looking for times when velocity is strictly negative.

© No, you won't have anything even close to this annoying on the test, derivative-wise. It's always a good idea to read footnotes, don't you think?



I'll look at what happens to the sign of  $v$  at a value in each of the three intervals I get. Since the first part of  $v$  is squared, it won't affect the sign, and I only have to consider the sign of  $8t - 11$ .

For values between 0 and 1,  $8t < 11$ , and  $v$  will be negative, which is why that's shaded on the number line. For values between 1 and  $1\frac{3}{8} = \frac{11}{8}$ ,  $8t < 11$ , so  $v$  is still negative, and that section is shaded. Finally, for values on the right end,  $8t > 11$ , and  $v$  is positive.

So  $v(t) < 0$  for  $t \in [0, 1) \cup (1, 1\frac{3}{8})$ .

- e) Find the acceleration of the particle at any time  $t \geq 0$ .  
 $a(t) = v'(t)$ . Using the product rule,  
 $a(t) = (t-1)^2 \cdot 8 + (8t-11) \cdot 2(t-1)^1 \cdot 1$   
 $= 2(t-1)(4t-4+8t-11)$   
 $= 2(t-1)(12t-15)$   
 $= 6(t-1)(4t-5)$

No, you didn't have to make this as pretty as I did.

- f) Find the value of  $t$  when the particle is moving and the acceleration is zero.

First, find the times when acceleration is zero.

$$a(t) = 6(t-1)(4t-5) = 0$$

$$t = 1 \text{ or } t = 1.25$$

But now I need to see if the particle is moving. That means its velocity has to be nonzero.

$$v(t) = (t-1)^2 (8t-11), \text{ so}$$

$$v(1) = (1-1)^2 (8 \cdot 1 - 11) = 0.$$

$$v(1.25) = (1.25-1)^2 (8 \cdot 1.25 - 11)$$

$$= 0.0625 \cdot -1 = -0.0625.$$

Therefore  $t = 1.25$  is the only time that satisfies the requirement.

True or false? Justify.

45. If  $f'(7) = 5$ , then  $f$  is continuous at  $x = 7$ .

**True.** Differentiability implies continuity.

46. If  $f'(7) = 5$ , then  $f$  is decreasing at  $x = 7$ .

**False.** The slope is positive, so the function is sloped up, and thus increasing.

47. If  $f'(7) = 5$ , then  $y - 5 = 5(x - 7)$  is tangent to  $f$  at the point where  $x = 7$ .

**False.** The slope is correct, as is the value of  $x$  at the point, but there is no reason to believe that the  $y$ -value there is 5.

48. If  $f'(7) = 5$ , then  $\lim_{\Delta x \rightarrow 0} \frac{f(5 + \Delta x) - f(5)}{\Delta x} = 7$ .

**False.** The 7 and the 5 are backwards. The limit really says that  $f'(5) = 7$ .

49. If  $f'(7) = 5$ , then  $\lim_{x \rightarrow 7} \frac{f(x) - f(7)}{x - 7} = 5$ .

**True.** It's an alternative definition of the derivative at  $x = 7$ .

50. If  $f'(7) = 5$ , then  $(f^{-1})'(7) = \frac{1}{5}$ .

**False.** It's definitely correct that the slopes of inverse functions are reciprocals of each other, but the  $x$ -value on  $f$  isn't (necessarily) the same as the  $x$ -value on  $f^{-1}$ . The formula for calculating the derivative of an inverse function is  $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ , so to find

$(f^{-1})'(7)$ , I'd need  $f^{-1}(7)$ . It's not here.

51.  $\lim_{x \rightarrow \pi} \left( \frac{\sin x}{1 - \cos x} \right) = \frac{\sin \pi}{1 - \cos \pi}$   
 $= \frac{0}{1 - (-1)} = \frac{0}{2} = 0$

Yeah, just plug it in. Don't be L'Hypnotized into thinking that every limit uses L'Hôpital's rule. First you've got to see if it is the right indeterminate form:  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

52.  $\lim_{x \rightarrow \infty} \left( \frac{5x^3 - 2x + 7}{3x^3 + 2x^2} \right)$

As  $x$  approaches infinity, both the numerator and denominator go to infinity. *This* one is a L'Hôpital problem.

$$\lim_{x \rightarrow \infty} \left( \frac{5x^3 - 2x + 7}{3x^3 + 2x^2} \right) = \lim_{x \rightarrow \infty} \left( \frac{15x^2 - 2}{9x^2 + 4x} \right). \text{ Now let}$$

$x$  go to infinity. Except that will be  $\infty$  over  $\infty$  again. Let's go L'Hô!

$$\lim_{x \rightarrow \infty} \left( \frac{15x^2 - 2}{9x^2 + 4x} \right) = \lim_{x \rightarrow \infty} \left( \frac{30x}{18x + 4} \right)$$

And once again, infinities. Once more will do it, though.

$$\lim_{x \rightarrow \infty} \left( \frac{30x}{18x+4} \right) = \lim_{x \rightarrow \infty} \left( \frac{30}{18} \right) = \frac{30}{18} = \frac{5}{3}$$

Could I have gotten that answer earlier on? Sure. As  $x$  gets huge, only the highest powers of  $x$  matter, and I get the same thing from the ratios of the leading terms. But that doesn't let me explain the use of L'Hôpital.

53.  $\lim_{x \rightarrow 0} \left( \frac{\tan x}{x + \sin x} \right)$

Plugging in 0 makes the fraction evaluate to  $\frac{0}{0}$ . Using L'Hôpital's rule gives...

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \frac{\tan x}{x + \sin x} \right) &= \lim_{x \rightarrow 0} \left( \frac{\sec^2 x}{1 + \cos x} \right) = \frac{\sec^2 0}{1 + \cos 0} \\ &= \frac{1^2}{1+1} = \frac{1}{2}. \end{aligned}$$

54.  $\lim_{x \rightarrow 0} \left( \frac{\tan x - x}{x^3} \right)$  evaluates to  $\frac{0}{0}$ , too. So use

L'Hôpital's rule:

$$\lim_{x \rightarrow 0} \left( \frac{\tan x - x}{x^3} \right) = \lim_{x \rightarrow 0} \left( \frac{\sec^2 x - 1}{3x^2} \right).$$

This time, substitution gets me

$$\frac{\sec^2 0 - 1}{3 \cdot 0^2} = \frac{1 - 1}{0} = \frac{0}{0}. \text{ So use it again:}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \frac{\sec^2 x - 1}{3x^2} \right) &= \lim_{x \rightarrow 0} \left( \frac{2 \sec x \cdot \sec x \tan x}{6x} \right) \\ &= \frac{2 \cdot \sec 0 \cdot \sec 0 \cdot \tan 0}{6 \cdot 0} = \frac{2 \cdot 1 \cdot 1 \cdot 0}{0} = \frac{0}{0}. \end{aligned}$$

One more time (or I'm giving up!):

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \frac{2 \sec^2 x \tan x}{6x} \right) &= \\ \lim_{x \rightarrow 0} \left( \frac{2 \sec^2 x \cdot \sec^2 x + \tan x \cdot 4 \sec x \cdot \sec x \tan x}{6} \right) &= \\ = \frac{2 \cdot \sec^4 0 + 4 \tan^2 0 \sec^2 0}{6} &= \\ = \frac{2 \cdot 1 + 4 \cdot 0 \cdot 1}{6} = \frac{2}{6} = \frac{1}{3}. \end{aligned}$$

Finally! That numerator was yuck. It's a product rule, with a chain rule for the  $\sec^2 x$ . Oh, the insanity!

(Could this have been any easier?)

Actually, yes. Using a trig identity,  $\sec^2 x - 1 = \tan^2 x$ . Then changing the whole thing into sines and cosines would let you use

the fact that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  to get the answer

with a lot less differentiation. But did *you* think of that? I thought not.)

55.  $\lim_{x \rightarrow 0} \left( \frac{\sin x - x}{x^3} \right)$  gives  $\frac{0}{0}$ . Here's your pal, L'Hôpital!

$$\lim_{x \rightarrow 0} \left( \frac{\cos x - 1}{3x^2} \right) \cdot \frac{0}{0} \text{ again!}$$

$$\lim_{x \rightarrow 0} \left( \frac{-\sin x}{6x} \right) \cdot \frac{0}{0}. \text{ Who knew?}$$

$$\lim_{x \rightarrow 0} \left( \frac{-\cos x}{6} \right) = -\frac{1}{6}.$$