Answers for Calculus Review, §3.5-3.8

1. $f(x) = (x^2 + 3)^{29}$
This is just straight chain rule; the derivative of stuff^{29} is $29 \cdot \text{stuff}^{28} \cdot d(\text{stuff})$.
$f'(x) = 29(x^2 + 3)^{28} \cdot 2x$
$\boxed{f'(x) = 58x(x^2 + 3)^{28}}$

2. This one is the same, with the exception of the fact that I'll need to rewrite the radical as a power first.
$y = \sqrt{2-3x^2} = (2 - 3x^2)^{1/2}$
$$\frac{dy}{dx} = \frac{1}{2}\left(2-3x^2\right)^{-1/2} \cdot -6x$$
$$= -3x(2 - 3x^2)^{-1/2}$$
$$\boxed{\frac{dy}{dx} = \frac{-3x}{\sqrt{2-3x^2}}}$$
Note that the answer on the line before the box is *also* completely correct.

3. $$y = \left(\frac{x-1}{x+1}\right)^2$$
Hey, more fun! The main thing going on here is stuff^2, which has a nice easy derivative. However, when we multiply by the derivative of the inside, we'll need the quotient rule.
$$\frac{dy}{dx} = 2\left(\frac{x-1}{x+2}\right)^1 \frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}$$
Clearly the numerator can clean up some.
$$= 2\left(\frac{x-1}{x+1}\right)\cdot\frac{x+1-x+1}{(x+1)^2}$$
$$= 2\cdot\frac{x-1}{x+1}\cdot\frac{2}{(x+1)^2}$$
$$\boxed{\frac{dy}{dx} = \frac{4(x-1)}{(x+1)^3}}$$
That's as nice as it gets.

4. $y = (\csc x + \cot x)^{-1}$
There are two different ways to deal with this. I could rewrite it as a fraction, or I could use the chain rule on the function as it is written. I choose the chain rule, since that's one of the things the quiz is testing. Notice the extensive use of trig derivatives that you should have memorized by now!
$$y' = -1(\csc x + \cot x)^{-2} \cdot (-\csc x \cot x - \csc^2 x)$$
$$= \frac{\csc x \cot x + \csc^2 x}{(\csc x + \cot x)^2}$$
I might stop there, but I think it will get better. Factoring out the csc x out of the top looks good.
$$= \frac{\csc x(\cot x + \csc x)}{(\csc x + \cot x)^2}$$
Hey, I can cancel! (Note that before I had factored, it would *not* be okay to cancel. That's not an optional step.)
$$\boxed{y' = \frac{\csc x}{\csc x + \cot x}}$$

5. $g(t) = \sin(3t^4)$
This is a nice straightforward chain rule again. Derivative of the outside function times derivative of the inside function. Notice that the inside of cosine is still $3t^4$, not the derivative yet.
$g'(t) = \cos(3t^4) \cdot 12t^3$
$\boxed{g'(t) = 12t^3 \cos(3t^4)}$

6. $h(x) = x^2 (4x^3 + 5)^6$
Now, that's nice. A product rule, where the second of the two factors is a composite function and will need the chain rule. The overall structure of the answer will be like this:
$h'(x) = \text{first} \cdot d(\text{second}) + \text{second} \cdot d(\text{first})$,
where d(second) uses the chain rule. Here it is.
$h'(x) = x^2 \cdot 6(4x^3 + 5)^5 \cdot 12x^2 + (4x^3 + 5)^6 \cdot 2x$
$= 2x(4x^3 + 5)^5(36x^3 + 4x^3 + 5)$
$$\boxed{\begin{aligned} h'(x) &= 2x(4x^3 + 5)^5(40x^3 + 5) \\ &= 10x(4x^3 + 5)^5(8x^3 + 1)\end{aligned}}$$
"What the heck?" you ask yourself. Well, I noticed after the first line of the derivative that the two parts both had a power of $(4x^3 + 5)$, so I factored out the lower power. They also had a common factor of $2x$, so that came out front as well. From the first part, I was left with none of the $4x^3 + 5$ terms, but there were still three x's and $3 \cdot 12 = 36$ as a coefficient. From the second part, all that was left was a $4x^3 + 5$. I'll leave it to you to decide what happened between the first and second lines inside the answer box.

7. $y = \cos^2(3x - 2) = [\cos(3x - 2)]^2$
We write trig functions with powers on the functions themselves as a shorthand notation. What we really *mean* is that we're raising the function to a power after evaluating it. In other words, we need parentheses:
$y = \cos^2(3x - 2) = [\cos(3x - 2)]^2$
Now the problem is a pretty straightforward chain rule, with the joyous addition of a *second* "inside" part. The first inside function is cosine; the second one is $3x - 2$. I'll break it down.

$$y' = 2[\cos(\text{stuff})]^1 \cdot \frac{d}{dx}\left[\cos\left(\textit{stuff}\right)\right]$$

$$= 2[\cos(3x-2)]^1 \cdot -\sin(\text{stuff}) \cdot \frac{d}{dx}\left[\textit{stuff}\right]$$

$$y' = 2[\cos(3x-2)]^1 \cdot -\sin(3x-2) \cdot 3$$

$$\boxed{y' = -6\sin(3x-2)\cos(3x-2)}$$

I put sine first because… because I'm biased toward sine, I guess. It's a long story, and has to do with cosine being sort of superior-acting in identities and such, and it's not at all important.

8. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$
Quotient rule this time, with a chain in taking the derivative of the denominator. For that part, I'll want to think of the denominator as $(1 + x^2)^{1/2}$. Notice that it's *not* a negative power, because I'm keeping the radical in the denominator to use the quotient rule.

$$f'(x) = \frac{\sqrt{1+x^2} \cdot 1 - x \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x}{\left(\sqrt{1+x^2}\right)^2}$$

$$f'(x) = \frac{\sqrt{1+x^2} - x^2 \cdot (1+x^2)^{-1/2}}{1+x^2}$$

In this case, of course, squaring off the radical *does* make sense. Watch what happens next. I'm going to factor out the lowest power of $(1 + x^2)$ from the numerator, which would be the –1/2 power.

$$= \frac{(1+x^2)^{-1/2}(1+x^2-x^2)}{1+x^2}$$

I was left with $1 + x^2$ for the first term, and only the x^2 for the second. How did I do that?

It's all about the exponents. When you divide powers of the same base, you subtract the exponents. In this case, in factoring out $(1 + x^2)^{-1/2}$, I had to subtract its exponent from the $(1 + x^2)^{1/2}$ that was already there. And $\frac{1}{2} - \left(-\frac{1}{2}\right) = 1$, the first power. I'll leave it to you to see why there was only an x^2 left from the second term.

Whew. Combining like terms leaves me with $\dfrac{\left(1+x^2\right)^{-1/2} \cdot 1}{1+x^2} = \dfrac{\left(1+x^2\right)^{-1/2}}{1+x^2}$. But wait! Aren't those both powers of the same base, too? That they are, young scholar. If we move the term from the numerator to the denominator, it will have a positive exponent, and adding those exponents gives this:

$$\boxed{f'(x) = \frac{1}{(1+x^2)^{3/2}}}$$

Yes. I'm done now. For those of you who are well into the freaking-out stage, stop it. You could have stopped with the second line that starts with f'. The prettying-up is mainly useful for (a) multiple choice answers, (b) ease of substituting values, and (c) taking a second derivative. Most of the time, you won't need to do it. But it *is* really nice, isn't it?

9. $f(\theta) = \tan^2\theta = (\tan\theta)^2$
Don't let the θ scare you; it acts just like x. Notice that I rewrote the second power on parentheses to make it easier to see how the chain rule will apply.

$$f'(\theta) = 2(\tan\theta)^1 \cdot \sec^2\theta$$

$$\boxed{f'(\theta) = 2\tan\theta\sec^2\theta}$$

Kind of a treat after the last one!

10. $r = \sqrt{\theta\sin\theta} = (\theta\sin\theta)^{1/2}$
Again, the variable isn't a problem. However, this time there's a product under the radical. The radical itself is a power, but to take the derivative of the radicand, I'll need the product rule. Here's an outline.

$d[\text{stuff}^{1/2}] = \frac{1}{2}\left(\text{stuff}\right)^{-1/2} \cdot d\left(\text{stuff}\right)$. And that last part is a nice little product rule. Watch.

$$\frac{dr}{d\theta} = \frac{1}{2}\left(\theta\sin\theta\right)^{-1/2} \cdot \left(\theta\cos\theta + \sin\theta \cdot 1\right)$$

$$\boxed{\frac{dr}{d\theta} = \frac{\theta\cos\theta + \sin\theta}{2\sqrt{\theta\sin\theta}}}$$

11. $y = \arctan x$.
You did memorize this, didn't you?

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

12. $y = \text{arcsec}(5x^4)$.
First of all, I'll need to know the derivative of the arcsecant function. Here's the rule:

$$\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{|x|\sqrt{x^2-1}}.$$

In this case, I'll also be using the chain rule, because the inside of the function is also a function that gets its own derivative.

$$\frac{dy}{dx} = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}} \cdot 4 \cdot 5x^3$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}} = \boxed{\frac{4}{x\sqrt{25x^8-1}}}.$$

How do I know it's okay to remove the absolute value? Because $5x^4$ is positive. Yep.

13. $y = \sin(\arccos x)$.
Chain, chain, chain…

$$\frac{dy}{dx} = \cos(\text{stuff})\frac{d}{dx}(\text{stuff})$$

$$= \cos(\arccos x) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= x \cdot \frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-x}{\sqrt{1-x^2}}}$$

Did you notice how the cosine of the arccosine of x did their inverse function thing to leave just x? Nice.

14. $y = x \arccos x - \sqrt{1-x^2}$.
The first part requires a product rule, and the second part is going to use the chain rule. I'll break it down:

$$\frac{d}{dx}[x \arccos x] = x \cdot \frac{d}{dx}[\arccos x] + \arccos x \cdot 1$$

$$= x \cdot \frac{-1}{\sqrt{1-x^2}} + \arccos x$$

$$= \frac{-x}{\sqrt{1-x^2}} + \arccos x$$

$$\frac{d}{dx}\left[\sqrt{1-x^2}\right] = \frac{d}{dx}\left[(1-x^2)^{1/2}\right]$$

$$= \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-2x}{2\sqrt{1-x^2}}$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

Putting it all together, the derivative of $y = x \arccos x - \sqrt{1-x^2}$ will be

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} + \arccos x - \left(\frac{-x}{\sqrt{1-x^2}}\right)$$

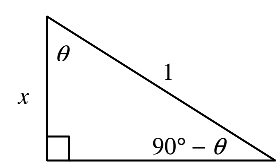
$$= \frac{-x}{\sqrt{1-x^2}} + \arccos x + \frac{x}{\sqrt{1-x^2}}$$

$$= \boxed{\arccos x}$$

15. Differentiate $y = \arcsin x + \arccos x$.
Easy!

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = \boxed{0}.$$

Zero. But if that's true, the original function must be a constant. And it is. It's $\frac{\pi}{2}$. Try thinking about it in terms of a right triangle. Here, I'll draw it, but the rest of the thinking is up to you.



The next six problems require implicit differentiation, where you take the derivative without first solving for y, using the chain rule to help with differentiating terms that contain y. Then you isolate the y'. You can write y' or $\frac{dy}{dx}$ for the derivative of y; I've mostly written y' because it's more compact.

16. Differentiate $\sqrt{xy} = 1$.
There are actually a couple of different ways to go about this. I'm going to first do this the straight-ahead way that most people would think of first. The square root is the same as a power of one-half. So $(xy)^{1/2} = 1$. For the derivative of the left side, I'll need the chain rule and the product rule.

$$\tfrac{1}{2}(xy)^{-1/2} \cdot \frac{d}{dx}(xy) = 0$$

$$\tfrac{1}{2}(xy)^{-1/2} \cdot (xy' + y \cdot 1) = 0$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{xy}}(xy'+y)=0$$

$$\frac{1}{2\sqrt{xy}}(xy'+y)=0$$

The y' is sort of stuck inside the parentheses here. I could distribute the fraction, but that is *not* looking attractive. I think I'll just multiply both sides by $2\sqrt{xy}$ to eliminate the fraction. If there were something other than zero on the right side, this would put the radical there; since it's zero, the unattractive radical just disappears. (Also, I know from the original problem that $\sqrt{xy}=1$, so I'm *really* just multiplying by 2.)

Now I have $xy' + y = 0$. I'll subtract the y and divide by the x, and that's it.

$$xy' = -y$$

$$\boxed{y'=-\frac{y}{x}=\frac{dy}{dx}}$$

You might recall that I mentioned that there was more than one way to go about this. Here's the slick way. If $\sqrt{xy}=1$, why not just square both sides to get an easier function? After all, if the principal square root of a number is 1, the number itself must be 1.

$$\sqrt{xy}=1$$

$$xy = 1$$

Here, I could either go with the product rule or solve for y and use the power rule. I think I'll do the second one, since, really, I've already done the product rule on this in the first version of the solution.

If $xy = 1$, then $y=\frac{1}{x}=x^{-1}$, and $\boxed{\frac{dy}{dx}=-1x^{-2}=-\frac{1}{x^2}}$. Is that really the same answer? Yep. Here's how I know.

$-\frac{1}{x^2}=-\frac{1}{x}\cdot\frac{1}{x}$, and $y=\frac{1}{x}$. Substituting, $-\frac{1}{x^2}=-\frac{1}{x}\cdot y=-\frac{y}{x}$. Same answer. Of course, you only have to do the problem one way. I thought some of you might appreciate the slickness of this one.

17. Given $x^3 + y^3 = 3xy$; differentiate both sides. The right side will require the product rule.

$$3x^2 + 3y^2y' = 3xy' + 3y$$

I'll divide by 3 to make the numbers easier to deal with.

$$x^2 + y^2y' = xy' + y$$

Then put both terms with y' on one side, and the other two terms on the other side.

$$y^2y' - xy' = y - x^2$$

Factor out y', then divide to isolate the derivative.

$$y'(y^2 - x) = y - x^2$$

$$y'=\boxed{\frac{dy}{dx}=\frac{y-x^2}{y^2-x}}$$

18. Given $2y = x^2 + \sin y$; differentiate both sides, remembering to multiply by y' when differentiating a term with y.

$$2y' = 2x + \cos y\cdot y'$$

Then rearrange terms to get the y' terms all on one side, factor it out, and divide to isolate the derivative.

$$2y' - \cos y\cdot y' = 2x$$

$$y'(2 - \cos y) = 2x$$

$$y'=\boxed{\frac{dy}{dx}=\frac{2x}{2-\cos y}}$$

19. Differentiate $1 - xy = x - y$.

I'll be using implicit differentiation again, although it *is* possible to solve for y if you should choose to. For the xy term, I'll be employing the product rule.

$$0 - (x\cdot y' + y\cdot 1) = 1 - y'$$

$$-xy' - y = 1 - y'$$

Now I'll rearrange things a little, adding y to bring it to the right side, and adding y' to get it to the left.

$$y' - xy' = 1 + y$$

Factor out the y', and divide.

$$y'(1 - x) = 1 + y$$

$$y'=\boxed{\frac{dy}{dx}=\frac{1+y}{1-x}}$$

You might wonder what would happen if you solved the original equation for y first, and then differentiated. Let's see.

$$1 - xy = x - y$$

$$1 - x = xy - y$$

$$1 - x = y(x - 1)$$

$$y=\frac{1-x}{x-1}.$$

Now, you could use the quotient rule here. (Yes, I know some of you noticed something else. Hold on a sec.)

$$\frac{dy}{dx}=\frac{(x-1)\cdot(-1)-(1-x)\cdot 1}{(x-1)^2}$$

$$=\frac{-x+1-1+x}{(x-1)^2}=\frac{\cancel{-x}\cancel{+1}\cancel{-1}+\cancel{x}}{(x-1)^2}$$

$$=\frac{0}{(x-1)^2}\boxed{=0}.$$

Yep, the derivative is zero. Those of you I told to hold on — that top line, $y=\frac{1-x}{x-1}$, *is* equivalent to –1, and you could differentiate that to get 0, too. You might find it a little troubling to think that the answer I got in the first place, $y'=\frac{dy}{dx}=\frac{1+y}{1-x}$, is really equal to 0. It *is*, and I'll show you why.

$$y'=\frac{dy}{dx}=\frac{1+y}{1-x} \text{ and } y=\frac{1-x}{x-1}$$

Substituting, $\frac{dy}{dx}=\frac{1+\frac{1-x}{x-1}}{1-x}=\frac{\frac{x-1}{x-1}+\frac{1-x}{x-1}}{1-x}$

$$=\frac{\frac{x-1+1-x}{x-1}}{1-x}=\frac{\frac{0}{x-1}}{1-x}=0.$$

Crazy.

20. Differentiate $\tan(x + y) = x$.

$$\frac{d}{dx}\left[\tan(x+y)\right]=\frac{d}{dx}[x]$$

$\sec^2(x + y) \cdot (1 + y') = 1$

To isolate the y', first divide by the $\sec^2$ term, and then subtract.

$$1+y'=\frac{1}{\sec^2(x+y)}=\cos^2(x+y)$$

$$y'=\boxed{\frac{dy}{dx}=\cos^2(x+y)-1}$$

21. Find $\frac{dy}{dx}$ for $y = \tan(xy)$.

Implicit one more time. Take the derivative of both sides. The right side will need the chain rule, and the product rule However, because I've done it more times than I care to recall at the moment, I'm going to steal the derivative of xy from problem 19.

$y' = \sec^2(xy) \cdot (xy' + y)$

Now it looks a little trickier than some of these. There a y' isolated on the left, and one that looks hopelessly entangled in the right side. The distributive property rides to the rescue! Multiply out the right-hand side. One of the terms will have y', and the other one won't. Then rearrange a little, factor, and divide. Here goes.

$y' = \sec^2(xy) \cdot xy' + \sec^2(xy) \cdot y$

$y' - \sec^2(xy) \cdot xy' = y \sec^2(xy)$

$y' (1 - x \sec^2(xy)) = y \sec^2(xy)$

$$y'=\boxed{\frac{dy}{dx}=\frac{y\sec^2(xy)}{1-x\sec^2(xy)}}.$$

Can I simplify this? I think so. After all, secant is the reciprocal of cosine. You don't *have* to simplify it, but what the heck.

$$\frac{y\sec^2(xy)}{1-x\sec^2(xy)}=\frac{1}{\cos^2(xy)}\cdot\frac{y}{1-x\sec^2(xy)}$$

$$\boxed{=\frac{y}{\cos^2(xy)-x}=\frac{dy}{dx}}.$$

How did the last part of the denominator disappear like that? It's because $\cos^2(xy)$ and $\sec^2(xy)$ are reciprocals, and therefore multiply to be one.

The last four questions ask for second derivatives, and in each case, they're functions for which I've already found the first derivatives above. I did that to save some typing and make things a little less tedious for you, too. On the quiz itself, there will be *one* problem for which you'll have to find a second derivative, and you will *not* already have found the first derivative. You'll have to do it twice. Life is hard.

22. $f(x) = (x^2 + 3)^{29}$

$f'(x) = 58x(x^2 + 3)^{28}$, from problem 1.

To take another derivative, I'll need the product rule, because now there's an x outside the parentheses.

$f''(x) = 58x \cdot 28(x^2 + 3)^{27} \cdot 2x + (x^2 + 3)^{28} \cdot 58$

Yes, I *am* going to simplify this. See all of the similarities in the two pieces?

Factoring out 58 and $(x^2 + 3)^{27}$ gives me this:

$f''(x) = 58(x^2 + 3)^{27}[x \cdot 28 \cdot 2x + (x^2 + 3)^1]$

$= 58(x^2 + 3)^{27}[56x^2 + x^2 + 3]$

$= 58(x^2 + 3)^{27}(57x^2 + 3)$

Now, I *could* stop here. But if I did, someone would doubtless point out to me that the last binomial has a common factor of 3 that can be taken out. So…

$f''(x) = 58(x^2 + 3)^{27} \cdot 3(19x^2 + 1)$

$= \boxed{174(x^2 + 3)^{27}(19x^2 + 1)}$

I'm done. And I did do the arithmetic in my head. However, I decided it would be wise to check my result with a calculator. I was right.

23. $g(t) = \sin(3t^4)$

$g'(t) = 12t^3 \cos(3t^4)$, from problem 5. To take a second derivative, it's the product rule yet again. (I'm bored. This is tedious. Aren't you glad there are only 8 questions total on the quiz?)

$g''(t) = 12t^3 \cdot -\sin(3t^4) \cdot 12t^3 + \cos(3t^4) \cdot 36t^2$

$= \boxed{-144t^6 \sin(3t^4) + 36t^2 \cos(3t^4)}$

No factoring. I don't want to, and in the end it wouldn't be "better," just different.

24. $y = \sqrt{2 - 3x^2}$

$\dfrac{dy}{dx} = \dfrac{-3x}{\sqrt{2-3x^2}}$, from problem 2. I don't know what I was thinking putting this many examples on. But here goes. For the second derivative, I'll need the quotient rule. I'm going to write the radical exclusively as a power.

$$\frac{d^2y}{dx^2} = \frac{\left(2-3x^2\right)^{1/2} \cdot -3 - (-3x) \cdot \frac{1}{2}\left(2-3x^2\right)^{-1/2} \cdot -6x}{\left(\left(2-3x^2\right)^{1/2}\right)^2}$$

$$= \frac{-3\left(2-3x^2\right)^{1/2} + 3x \cdot \left(2-3x^2\right)^{-1/2} \cdot -3x}{2-3x^2}$$

(The $-3x$ at the end is from $1/2 \cdot -6x$.)

$$= \frac{-3\left(2-3x^2\right)^{1/2} - 9x^2\left(2-3x^2\right)^{-1/2}}{2-3x^2}.$$

As a mathochist (I made that up; what do you think?), I'll factor some. It's like #8.

$$= \frac{-3\left(2-3x^2\right)^{-1/2}\left[\left(2-3x^2\right)^1 + 3x^2\right]}{2-3x^2}$$

$$= \frac{-3\left(2-3x^2\right)^{-1/2} \cdot 2}{2-3x^2} = \boxed{\frac{-6}{\left(2-3x^2\right)^{3/2}}}$$

25. $f(\theta) = \tan^2 \theta$

Last question!!!

$f'(\theta) = 2 \tan \theta \sec^2 \theta$ from problem 9. To differentiate it, I'll once again use the product rule. (You know, it's not that surprising. The chain rule involves multiplication, right? Having a second derivative need the product rule actually makes a lot of sense.)

$f''(\theta) = 2 \tan \theta \cdot 2(\sec \theta)^1 \cdot \sec \theta \tan \theta$
$\qquad + \sec^2 \theta \cdot 2 \sec^2 \theta$

$= 4 \tan^2 \theta \sec^2 \theta + 2 \sec^2 \theta$

For the sake of beauty…

$\boxed{f''(\theta) = 2 \sec^2 \theta (2 \tan^2 \theta + 1)}$

By the way, throughout this review sheet, you'll notice my excellent use of notation for derivatives. I did that on purpose. If you've found any mistakes, let me know (mailto:bhsfrisbie@earthlink.net).